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Bifurcation of positive solutions for a three-species food chain model with diffusion

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ABSTRACT

In this paper, we consider a reaction–diffusion system describing a three-species Lotka–Volterra food chain model with homogeneous Dirichlet boundary conditions. By regarding the birth rate of prey r_1 as a bifurcation parameter, the global bifurcation of positive steady-state solutions from the semi-trivial solution set is obtained via the bifurcation theory. The results show that if the birth rate of mid-level predator and top predator are located in the regions $0 < r_2 < \lambda_1\left(a_{23}u_{3r_3}\right)$ and $r_3 > \lambda_1$, respectively. Then the three species can co-exist provided the birth rate of prey exceeds a critical value. Moreover, an explicit expression of coexistence steady-state solutions is constructed by applying the implicit function theorem. It is demonstrated that the explicit coexistence steady-state solutions is locally asymptotically stable.

1. Introduction

Ecological systems are characterized by the interactions of different species within a fluctuating natural environment. In particular, predator–prey type interaction is one of the basic interspecies relations in biology and ecology, and it is also the basic block of the complicated food chain, food web and biochemical network structure. Food chain models, as one of the most important predator–prey systems, have been studied on both spatially homogeneous situations [1–4] and spatially inhomogeneous situations [5–8] for last two decades. It is known in literature that the dynamics of the three-species model is much more complicated than that of the two-species model in a relative sense. Even for the ODE system, the dynamic behavior of positive solutions can be very complicated (see [1]). Consequently, multiple-species food chain models will continue to be one of dominant themes in both ecology and mathematical ecology due to its universal existence and importance.

In this paper, we consider the following Lotka–Volterra food chain model

$$\begin{cases} u_{1t} - \Delta u_1 = u_1\left(r_1 - a_{11}u_1 - a_{12}u_2\right), \\ u_{2t} - \Delta u_2 = u_2\left(r_2 + a_{21}u_1 - a_{22}u_2 - a_{23}u_3\right), \\ u_{3t} - \Delta u_3 = u_3\left(r_3 + a_{32}u_2 - a_{33}u_3\right) \quad \text{in } \Omega \times (0, \infty) \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N = 1, 2, 3, \ldots)$ with smooth boundary $\partial\Omega$, Δ stands for the Laplacian operator. r_j, a_{jj}, $j = 1, 2, 3$, a_{12}, a_{21}, a_{23}, a_{32} are all positive constants. u_j, $j = 1, 2, 3$, represents the population density of prey, mid-level predator and top predator species, respectively. r_j is the birth rate of the prey, mid-level predator and top predator,

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respectively; a_{jj} measures the intra-specific competition of the prey, mid-level predator and top predator, respectively; a_{12} and a_{23} denote the predation rate of per capita of the mid-level predator and top predator, respectively; a_{21} and a_{32} represent the conversion rate of the prey to the mid-level predator and the mid-level predator to the top one, respectively. Under homogeneous Neumann boundary conditions, we point out that by applying the results in [9,10], diffusion driven instability is not possible. Moreover, by constructing a Lyapunov functional, Xie [11] proved that the unique constant positive steady-state solutions is globally asymptotically stable even if u_1, u_2 and u_3 possess different diffusion coefficients, which indicated that no spatiotemporal patterns happen in system (1.1). Whereas, if some proper cross-diffusion terms are included in system (1.1), Ma et al. [12] studied the existence of non-constant positive steady-states as well as the Hopf bifurcation. They found that cross-diffusion can create not only stationary patterns but also spatially inhomogeneous periodic oscillatory patterns.

In this work we shall assume that the boundary is hostile and hence no individuals would choose to leave there and consequently, we shall subsequently consider homogeneous Dirichlet boundary conditions:

$$u_1 = u_2 = u_3 = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{1.2}$$

The problem of positive solutions for single and two interacting species with Dirichlet boundary conditions can be traced back to the late 70s and early 80s. An important early discovery is that the positive co-existence for two interacting species is determined by the spectral properties and the distribution of the equilibria of the system, see for instance [13–17]. Since then, existence, multiplicity, uniqueness and stability of positive solutions for various forms of two-species systems have received considerable attention in recent years (see, [18–27]). However, there is little concern on the problem of three-species systems, to our knowledge, only some particular cases have been studied [28–30]. Since the mathematical theory for three-species elliptic systems has not been well developed. As mentioned earlier, only non-trivial constant stationary solutions exist for system (1.1) with Neumann boundary conditions. While the boundary conditions change from homogeneous Neumann to Dirichlet, non-trivial constant stationary solutions disappeared. It is natural to ask whether non-constant steady-state solutions exist. The main task of this paper is to study this problem. Thus we will concentrate on the following elliptic system

$$\begin{cases} -\Delta u_1 = u_1\left(r_1 - a_{11}u_1 - a_{12}u_2\right), \\ -\Delta u_2 = u_2\left(r_2 + a_{21}u_1 - a_{22}u_2 - a_{23}u_3\right), \\ -\Delta u_3 = u_3\left(r_3 + a_{32}u_2 - a_{33}u_3\right) \quad \text{in } \Omega, \\ u_1 = u_2 = u_3 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Our analysis uses the idea developed in [13,31] and is based on the bifurcation theory, the implicit function theorem and the theory of space decomposition [32].

The structure of this paper is arranged as follows. In Section 2, we give some known results which are required later. By regarding r_1 as a bifurcation parameter, in Section 3, we discuss the bifurcation solutions of (1.3) from the semi-trivial solution set. In Section 4, the bifurcation from a three multiple eigenvalue is discussed. We end with concluding remarks in Section 5.

2. Preliminaries

Let $C_0(\overline{\Omega})$ be the usual Banach space of continuous functions on $\overline{\Omega}$ whose values on $\partial\Omega$ are zero, and $\|\cdot\|$ denote the maximum norm. $C_0^+(\overline{\Omega}) = \left\{u \in C_0(\overline{\Omega}) : u(x) \geq 0 \text{ for } x \in \overline{\Omega}\right\}$, $E = \left[C_0(\overline{\Omega})\right]^3$, $H = \left[C_0^+(\overline{\Omega})\right]^3$ and $X = \left[C^1(\overline{\Omega})\right]^3 \cap H$, where $C^1(\overline{\Omega})$ is the set of all continuous functions on $\overline{\Omega}$ whose partial derivatives up to the first order are continuous. We use standard notation $L^2(\overline{\Omega})$ and $H_0^2(\overline{\Omega})$ for the real-valued Sobolev spaces.

Let $\lambda_1(q)$ be the principal eigenvalue of the following problem

$$\begin{cases} -\Delta u + q(x)u = \lambda u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $q(x)$ is a smooth function from Ω to $\mathbb{R}$. It is easy to see that $\lambda_1(q)$ is an increasing function of q. Let $\lambda_1(0) = \lambda_1$, the eigenfunction corresponding to λ_1 is denoted by φ_1 which is normalized by $\max\varphi_1 = 1$. For the nonlinear boundary value problem

$$\begin{cases} -\Delta u + q(x)u = au - a_1u^2 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where q is as above and a and a_1 are real number with $a_1 > 0$. It is well known from [33] that if $a \leq \lambda_1(q)$, $u = 0$ is the unique nonnegative solution of (2.1); whereas if $a > \lambda_1(q)$, (2.1) has a unique positive solution. Suppose that $q(x) \equiv 0$, we denote the unique positive solution of

$$\begin{cases} -\Delta u = au - a_1u^2 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

by u_a, where $a > \lambda_1$. It is easy to prove that the mapping: $a \to u_a$ is strictly increasing and continuously differentiable in $a \in (\lambda_1, +\infty)$, and we have