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Two dimensional mixed finite element approximations for elliptic problems with enhanced accuracy for the potential and flux divergence

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A B S T R A C T

The purpose of the present paper is to analyse two new different possibilities of choosing balanced pairs of approximation spaces for dual (flux) and primal (potential) variables, one for triangles and the other one for quadrilateral elements, to be used in discrete versions of the mixed finite element method for elliptic problems. They can be interpreted as enriched versions of $BDFM_{k+1}$ spaces based on triangles, and of RT_k spaces for quadrilateral elements. The new flux approximations are incremented with properly chosen internal shape functions (with vanishing normal components on the edges) of degree $k+2$, and matching primal functions of degree $k+1$ (higher than the border fluxes, which are kept of degree k). In all these cases, the divergence of the flux space coincide with the primal approximation space on the master element, producing stable simulations. Using static condensation, the global condensed system to be solved in the enriched cases has same dimension (and structure) of the original ones, which is proportional to the space dimension of the border fluxes for each element geometry. Measuring the errors with L^2-norms, the enriched space configurations give higher convergence rate of order $k+2$ for the primal variable, while keeping the order $k+1$ for the flux. For affine meshes, the divergence errors have the same improved accuracy rate as for the error in the primal variable. For quadrilateral non-affine meshes, for instance trapezoidal elements, the divergence error has order $k+1$, one unit more than the order k occurring for RT_k spaces on this kind of deformed meshes. This fact also holds for ABF_k elements, but for them the potential order of accuracy does not improve, keeping order $k+1$.

1. Introduction

Mixed finite element methods have the ability to provide accurate and locally conservative fluxes, an advantage over standard H^1-conforming finite element discretizations [1]. They are based on simultaneous approximations of the primal

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(potential) and dual (flux) variables, involving two kinds of approximation spaces. In addition to $\mathbf{H}(\mathrm{div})$-conforming approximation spaces for the flux variable, with continuous normal components over element interfaces, the primal variable is usually represented in discontinuous finite element spaces.

Since the pioneering work by Raviart and Thomas [2] in 1977, different constructions of $\mathbf{H}(\mathrm{div})$-conforming approximation spaces have been proposed, e.g. in [3–5]. Recently, several other papers have appeared in the literature due to increasing interest on this subject [6–11]. In some contexts the vector basis functions are constructed directly on the physical element, but when it comes to solving practical problems in complex domains, shape functions are defined on the master element and then they are transformed to the elements of the partition using Piola transformations.

The main purpose of this article is to analyse two different ways of choosing balanced pairs of approximation spaces $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ and $U_h \subset L^2(\Omega)$, for flux and primal variables, by enriching the classic $BDFM_{k+1}$ spaces for triangular partitions $\mathcal{T}_h = \{K\}$ of the domain Ω, and RT_k spaces for quadrilaterals. When applied to discrete versions of the mixed finite element method for two dimensional elliptic problems, the goal is to obtain enhanced accuracy for the potential and flux divergence, while keeping fixed the flux accuracy. Using static condensation, the global condensed system to be solved in the enriched cases has same dimension (and structure) of the original ones, which is proportional to the space dimension of the border fluxes for each element geometry.

The different contexts share the following basic characteristics.

1. The functions $\mathbf{q} \in \mathbf{V}_h$ and $\varphi \in U_h$ are piecewise defined in each element K by locally backtracking polynomial spaces $\hat{\mathbf{V}}$ and $\hat{U}$ defined on the master element $\hat{K}$. The transformations $\mathbb{F}^{\mathrm{div}}$ and $\mathbb{F}$ used in their constructions are defined in terms of invertible linear or bilinear geometric mappings $\mathbf{x} : \hat{K} \to K$.
2. On the master element, the flux approximation spaces $\hat{\mathbf{V}}$ are spanned by a hierarchy of vector shape functions, which are organized into two classes: the shape functions of interior type, with vanishing normal components over all element edges, and the shape functions associated to the element edges. Thus, a direct factorization $\hat{\mathbf{V}} = \hat{\mathbf{V}}^{\partial} \oplus \mathring{\hat{\mathbf{V}}}$, in terms of edge and internal flux functions, naturally occurs.
3. In all the cases, the commutation de Rham property holds. Specifically,
$$\nabla \cdot \hat{\mathbf{V}} = \hat{U}. \tag{1}$$
4. In the flux space factorization of the enriched context, the edge component is kept fixed, but the internal component is formed by the original internal functions with degree increased by one. For consistency, the degree of the enriched scalar approximation space is also raised by one.

Different techniques have been used by numerical analysts to improve certain property of the numerical solution of a given method implemented on their codes. One popular strategy is to do post-processing, which has been considered for mixed finite element approximations of elliptic problems, as described in [12] and frequently adopted since then. The type of intrinsic enrichment strategy used in the present paper has also been previously adopted for the mixed method. For instance, on a triangular geometry, the classic BDM_k space configuration [3] uses vector polynomials of total degree k for the flux variable and polynomials of total degree $k-1$ for the potential variable, given uneven accuracy orders $k+1$ for the flux and k for the potential. The $BDFM_{k+1}$ space has been created as an enrichment of the BDM_k space configuration by using polynomials of total degree k for the potential variable, and for the flux variable the vector polynomials of total degree k are augmented with the internal shape functions of total degree $k+1$ to obtain the same accuracy order $k+1$ for both flux and potential variables. A similar principle has been applied on the construction of the RT_k space for quadrilateral elements [2]. The approximation space for the potential variable uses polynomials of maximum degree k in each coordinate, and for the flux approximation the vector polynomials of maximum degree k are augmented with some properly chosen internal shape functions of maximum degree $k+1$ in order to get stability. Analogously, to improve the divergence accuracy of the RT_k solutions based on non-affine quadrilateral meshes from order k to order $k+1$, the idea behind the construction of the ABF_k space in [6] is also to do enrichment by adding to the RT_k flux space some properly chosen internal functions of degree $k+1$, and by enlarging the potential space. All these constructions are based on the de Rham principle (i.e. divergence of the flux space coinciding with the potential space on the master element).

Following the same principle, enriched space versions $BDFM_{k+1}^{+}$ and RT_k^{+} are considered in the present paper, where their flux approximations are incremented with properly chosen internal shape functions of degree $k+2$, and matching primal functions of degree $k+1$ (higher than the border fluxes, which are kept of degree k). Measuring the errors with L^2-norms, an analysis is performed to demonstrate that for the flux variable the order of accuracy is $k+1$, for all space configurations. For the potential variable, the convergence rate is improved to order $k+2$ when the enriched versions are applied on triangles and quadrilaterals, as compared with order $k+1$ of the original ones $BDFM_{k+1}$ and RT_k, with same condensed matrix dimension (and structure). Furthermore, the divergence errors have the same orders of convergence reached by the potential variable, except when non-affine quadrilateral meshes are used.

For distorted quadrilateral meshes, it is well known that the divergence error is of order k for the classic RT_k space. We prove that this result can be upgraded to get divergence errors of order $k+1$ when the enriched version RT_k^{+} is applied. The error analysis follows the principles used for the construction of ABF_k spaces [6], with the advantage of getting an additional enhancement of the potential accuracy of order $k+2$ (instead of order $k+1$ given by the RT_k and ABF_k space configurations for quadrilateral meshes).

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