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Peter Monk and inverse scattering theory ☆

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ABSTRACT

Peter Monk has made numerous significant contributions to the field of inverse scattering theory. In the following I try to highlight Peter's most significant achievements in this area with emphasis on the renaissance that took place in the mathematical and numerical treatment of inverse scattering problems that began in the mid 1980s.

Peter Monk and I began working together on inverse scattering problems in the early 1980s. At that time the realization that inverse scattering problems were in general nonlinear and ill posed was just beginning to be accepted as the major issues facing a successful numerical solution of such problems and, having accepted this reality, efforts were focused on various versions of Newton's method to solve the nonlinear optimization problems that arose in such a study. The drawback of such an approach was that the implementation of Newton's method required the solution of a direct scattering problem at each step of the iteration process and for multi-dimensional scattering problems this became prohibitively expensive. This motivated Peter and I to look for a method that avoided the need to solve a series of direct scattering problems.

We began by considering the scattering of incident time harmonic acoustic plane waves by a sound–soft obstacle D, i.e. if u^s is the scattered field and u is the total field then

$$\begin{aligned} &\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus D \\ &u(x) = \exp(ikx \cdot d) + u^s(x) \\ &u = 0 \quad \text{on } \partial D \\ &\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^3$, $r = |x|$, $k > 0$ is the wave number, d is a unit vector giving the direction of propagation of the incident plane wave and D is assumed to be a bounded domain with smooth boundary ∂D such that $\mathbb{R}^3 \setminus D$ is connected. It can easily be shown that if u^s satisfies (1) then u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r}\left\{u_\infty(\hat{x}, d) + O\left(\frac{1}{r}\right)\right\} \tag{2}$$

where $\hat{x} = x/|x|$ and u_∞ is the *far field pattern* at the scattered field u^s where, since k is assumed to be fixed, we have suppressed the dependence of u_∞ on k. The *inverse scattering problem* is to determine D from u_∞ for $\hat{x}, d \in S^2$ where S^2 is the unit sphere in $\mathbb{R}^3$.

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The starting point of Peter's and my method for solving the inverse scattering problem was to consider the *far field operator* $\mathrm{F}: L^2(S^2) \to L^2(S^2)$ defined by

$$(\mathrm{F}g)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d) ds(d). \tag{3}$$

It was previously shown by Andreas Kirsch and myself [1] that F is injective with dense range if and only if there does not exist a Dirichlet eigenfunction for D corresponding to the eigenvalue k^2 which is a *Herglotz wave function*, i.e. a solution of the Helmholtz equation of the form

$$v(x) = \int_{S^2} e^{ikx\cdot d} g(d) ds(d), \quad x \in \mathbb{R}^3. \tag{4}$$

The function $g \in L^2(S^2)$ is called the *kernel* of the Herglotz wave function. Assuming that k^2 is not a Dirichlet eigenvalue and that D contains the origin, Peter and I then considered the *far field equation*

$$(\mathrm{F}g)(\hat{x}) = 1 \tag{5}$$

and, noting that 1 is the far field pattern of $\frac{1}{r}e^{ikr}$, it is easy to see that if $g \in L^2(S^2)$ satisfies (5) then the Herglotz wave function (4) satisfies

$$\begin{aligned} &\Delta v + k^2 v = 0 \quad \text{in } D \\ &v = -\frac{1}{r} e^{ikr} \quad \text{on } \partial D. \end{aligned} \tag{6}$$

From this we arrived at the optimization problem of using regularization methods to solve (5) for g and then finding D such that the Herglotz wave function v satisfies (6). Since it is no longer necessary to solve a direct scattering problem at each step of the optimization procedure, this method for finding D from u_∞ is relatively cheap and led to the first numerical solution of the three dimensional inverse scattering problem for acoustic waves for frequencies in the resonance region [2].

Having developed our method (which we called the *dual space method*) for solving the inverse scattering problem for an impenetrable obstacle, Peter and I then considered scattering by a penetrable inhomogeneous medium, i.e. the total field u satisfies

$$\begin{aligned} &\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3 \\ &u(x) = \exp(ikx \cdot d) + u^s(x) \\ &\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \end{aligned} \tag{7}$$

where $n(x)$ is positive, piecewise continuous and $m(x) := 1 - n(x)$ has compact support $\overline{D}$ such that D is simply connected, contains the origin and has smooth boundary ∂D with unit outward normal ν. The scattered field u^s again has the asymptotic behavior (2) and we can again define the far field operator by (3). In this case it was shown by Kirsch [3] that F is injective with dense range if and only if k is not a *transmission eigenvalue* (i.e. k is such that there exists a nontrivial solution of

$$\begin{aligned} &\Delta w + k^2 n(x) w = 0 \quad \text{in } D \\ &\Delta v + k^2 v = 0 \quad \text{in } D \\ &w = v \quad \text{on } \partial D \\ &\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D, \end{aligned} \tag{8}$$

where $w - v \in H_0^2(D)$) and in (8) v is a Herglotz wave function (the name transmission eigenvalue was given by Peter and myself in [4]). Then, as in the case of obstacle scattering, we can formulate the "dual space method" for solving the inverse scattering problem for inhomogeneous media by first using regularization methods to solve the far field equation (5) for g and then using optimization methods to determine $w(x)$ and $n(x)$ from (8) where v is the Herglotz wave function defined by (4) [4].

The above use of transmission eigenvalues in inverse scattering theory, together with the unusual nature of the transmission eigenvalue problem (8), has inspired a flood of papers on this subject which are far too numerous to describe here. For the interested reader we refer to the recent monograph [5] for which the general theory and further references can be found. Peter and I also wrote a number of papers on the dual space method in which the assumption what k^2 was a Dirichlet eigenvalue or k a transmission eigenvalue is removed and I will not comment further on these papers except to refer to [6] for a discussion and references.

In 1995 Andreas Kirsch was waiting for a flight to Germany at JFK International Airport in New York City after visiting Peter and me in Delaware. To pass the time away he looked at the far field equation (5) and the corresponding interior boundary value problem (6) and decided to change the origin, i.e. to replace (5) by

$$(\mathrm{F}g)(\hat{x}) = \Phi_\infty(\hat{x}, z) \tag{9}$$