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Convergence of Krylov subspace solvers with Schwarz preconditioner for the exterior Maxwell problem

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Dedicated to Professor Peter Monk on the occasion of his 60th birthday!

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ABSTRACT

The consideration of an integral representation as an exact boundary condition for the finite element resolution of wave propagation problems in exterior domain induces algorithmic difficulties. In this paper, we are interested in the resolution of an exterior Maxwell problem in 3D. As a first step, we focus on the justification of an algorithm described in literature, using an interpretation as a Schwarz method. The study of the convergence indicates that it depends significantly on the thickness of the domain of computation. This analysis suggests the use of the finite element term of Schwarz method as a preconditioner for use of Krylov iterative solvers. An analytical study of the case of a spherical perfect conductor indicates the efficiency of such approach. The consideration of the preconditioner suggested by the Schwarz method leads to a superlinear convergence of the GMRES predicted by the analytical study and verified numerically.

0. Introduction

We are interested in the resolution of the 3D exterior time-harmonic Maxwell equations. The problem consists in determining the field diffracted by an obstacle. To solve the equations posed in an unbounded domain, we introduce a fictitious boundary with an artificial boundary condition which models the infinity. In the literature, methods based on a local transparent boundary condition are often considered. They consist in an approximation of the Sommerfeld condition. In this context, we can consider a radiation condition at a finite distance [1] or the Bayliss–Gunzburger–Turkel-like conditions [2–5]. To obtain satisfying results, the artificial boundary should be chosen far from the boundary of the obstacle. Hence, the computational domain becomes large which increases computation and memory cost. An alternative to local transparent boundary conditions is the Perfectly Matched Layer (PML) method. It consists in replacing the artificial boundary by an absorbing layer of finite elements which vanishes the reflection. The method was introduced by Berenger and derived for the absorption of electromagnetic waves in [6].

In our paper, we rather consider an exact condition defined by an integral representation. The chosen strategy is named "coupling of finite elements and integral representation" and is designated by the acronym CEFRI for the French designation "Couplage Eléments Finis et Représentation Intégrale". CEFRI was initiated in [7] for hydrodynamic problems, and was presented and mathematically studied in the context of electromagnetism equations initially in [8]. In [9,10], a careful study of the numerical behavior of the formulation is presented for 2D Helmholtz equation and 3D Maxwell's equations, based on numerous numerical tests. [9] also includes a study of the convergence of the iterative resolution for the 2D Helmholtz

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equation. Such an exact condition defined by an integral representation has been successfully applied to the context of Ultra-Weak-Variational Formulation, which is an alternative to the finite elements, where the strategy was coupled to a Fast Multipole Method [11].

Like the consideration of a local transparent boundary condition, CEFRI leads to an equivalent problem posed on a reduced bounded domain delimited by the boundary of the scatterer and the artificial boundary, where the artificial boundary condition is expressed thanks to an integral representation of the unknown on this boundary. Even if integral operators are involved, no singularity occurs because the unknown on the artificial boundary is expressed using the unknown on the boundary of the scatterer. With such an approach, no *a priori* condition is required on the distance between the scatterer and the artificial boundary. The main difficulty consists in the elaboration of a numerical scheme for the resolution because of the integral operators which disturb the usual properties of the discrete equations. In [12], Jin and Liu solved the discrete system by a Jacobi method in order to avoid the inversion of the integral operators. The scheme can be interpreted as a Schwarz method with total overlap. This identification has been initially considered for the Poisson and Helmholtz exterior problems in [13,14]. In our paper, the interpretation is extended to the analytical exploration of the rate of convergence of the resolution strategy for Maxwell's equations. The theoretical analysis of the case of a spherical scatterer indicates that the method converges conditionally on the distance between the scatterer and the artificial boundary. Consequently, using the Jacobi scheme by Jin and Liu, the convergence may fail. However, this study of Schwarz method justifies the use of Krylov solvers and the choice of the preconditioner. We then analytically explore, in this paper, the convergence of a preconditioned GMRES for the resolution of the discrete system and prove the superlinear convergence of the method in the case of a spherical configuration. In [15], Jin and Liu also explored the use of a Krylov solver preconditioned by another formulation of the exterior problem to be solved. Their results are quite similar to ours but the preconditioner is different and consists of more components. In order to verify numerically our theoretical statements, we implemented the resolution strategies using the Finite Element library MÉLINA++ [16] which does not provide Nédélec elements. For the numerical results, we then considered the regularized Maxwell equations but the mathematical study of convergence of Jin and Liu algorithm is done for both the classical and the regularized Maxwell equations.

In next section, we introduce the physical problem and explain the application of CEFRI. Section 2 is devoted to the Schwarz interpretation of the resolution strategy suggested in [12]. This consideration enables us to estimate, in Section 3, the speed of convergence of the resolution algorithm in the case of a spherical scatterer using Jin and Liu algorithm. Some test-cases illustrate the theoretical estimation. In Section 4, we investigate the convergence of a Krylov method, the GMRES, combined to the preconditioner suggested by the previous analysis. Last section provides some numerical simulations which illustrate the convergence properties of this preconditioned application of the GMRES to CEFRI.

1. Scattering by a perfect conductor

Let us consider Ω_i a bounded scatterer in $\mathbb{R}^3$ with Lipschitz-continuous boundary Γ and Ω_e its unbounded complementary. We are concerned with the scattering of a time-harmonic electromagnetic wave by the perfect conductor Ω_i. Our purpose is to determine the total field $E = E^s + E^{inc}$ where E^{inc} is the incident wave and E^s is the scattered field. We then consider the following scattering problem with essential boundary condition on Γ and radiation condition at infinity:

$$\begin{cases} \operatorname{curl}\operatorname{curl} E - k_s^2 E = 0 \text{ in } \Omega_e, \\ E \times n_e = 0 \text{ on } \Gamma, \\ \displaystyle\lim_{R\to\infty} \int_{\|x\|=R} \|\operatorname{curl} E^s \times n_e - ik_s E^s\|^2 d\gamma = 0, \end{cases} \tag{1}$$

where k_s is the wavenumber and n_e is the exterior unit normal. In order to use standard Lagrange finite elements for the numerical resolution, we consider an equivalent elliptic problem by adding a regularizing grad–div term in the time-harmonic Maxwell equations, as described and justified in [8]. Problem (1) is then equivalent to the following one:

$$\begin{cases} \operatorname{curl}\operatorname{curl} E - t^{-1}\nabla(\operatorname{div} E) - k_s^2 E = 0 \text{ in } \Omega_e, \\ E \times n_e = 0, \, t^{-1}\operatorname{div} E = 0 \text{ on } \Gamma, \\ \displaystyle\lim_{R\to\infty} \int_{\|x\|=R} \|\operatorname{curl} E^s \times n_e - ik_s n_e \times (E^s \times n_e)\|^2 d\gamma = 0, \\ \displaystyle\lim_{R\to\infty} \int_{\|x\|=R} \left|\sqrt{t^{-1}}\operatorname{div} E^s - ik_s E^s \cdot n_e\right|^2 d\gamma = 0, \end{cases} \tag{2}$$

where the regularization term $t^{-1}\nabla(\operatorname{div} E)$ allows the use of a Galerkin finite element method (see [8]) and the regularization parameter t^{-1} depends on the permittivity and the permeability of the air. The parameter t is chosen positive. The choice $t = \infty$ corresponds to the initial scattering problem (1). Many different methods have been developed to solve the time-harmonic Maxwell equations in exterior domains. In this paper, we consider the coupling between finite elements and integral representation introduced by Hazard and Lenoir in [8]. The idea consists in defining an exact boundary condition on