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A P4 bubble enriched P3 divergence-free finite element on triangular grids

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Dedicated to Professor Peter Monk on the occasion of his 60th birthday.

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ABSTRACT

On triangular grids, the continuous P_k plus discontinuous P_{k-1} mixed finite element is stable for polynomial degree $k \geq 4$. When $k = 3$, the inf–sup condition fails and the mixed finite element converges at an order that is two orders lower than the optimal order. We enrich the continuous P_3 by adding some P_4 divergence-free bubble functions, to be exact, one P_4 divergence-free bubble function each component each edge. We show that such an enriched P_3–P_2 mixed element is inf–sup stable, and converges at the optimal order. Numerical tests are presented, comparing the new element with the P_4–P_3 element and the unstable P_3–P_2 element.

1. Introduction

It is a challenge to construct stable H^1 conforming mixed finite elements satisfying the incompressible condition exactly, in computing the Stokes or Navier–Stokes equations. That is, the velocity is approximated by the continuous piecewise polynomials of degree k and the pressure is approximated by the discontinuous piecewise polynomials of one degree less. Here the method is truly conforming in the sense that the finite element velocity is the H^1 projection of the true solution in a polynomial subspace. A breakthrough on the method was done by Scott and Vogelius in 1985 [1,2] that the method is stable and consequently of the optimal order of convergence on 2D triangular grids, for the P_k–P_{k-1} element, if $k \geq 4$, a magic number. What is this magic number k in 3D? Or if there is such a magic number in 3D? The problem remains open, after it was posted explicitly for so many years [1].

Scott and Vogelius showed that the P_k–P_{k-1} element is not stable on general triangular grids, if $k < 4$. However, on special triangular grids, low order elements may be stable. On Hsieh–Clough–Tocher macro-element grids, where each base triangle is split into 3 triangles by connecting the barycenter with three vertices, the P_2–P_1 and the P_3–P_2 mixed elements are stable, cf. [3–5]. On Powell–Sabin macro-element grids, where each triangle is split into 6 sub-triangles, even the P_1–P_0(subspace) element is stable [6]. When enriching the continuous P_k velocity space by some rational functions, Guzman and Neilan showed the enriched P_k–P_{k-1} element is stable for all $k \geq 1$, [7,8]. With additional continuity constraints, Falk and Neilan showed that the P_k–P_{k-1} element is stable if the continuous P_k velocity is also C_1 at vertices and the discontinuous P_{k-1} pressure is C_0 at vertices, cf. [9].

In 3D, the P_k–P_{k-1} mixed element is stable for all $k \geq 3$ on the Hsieh–Clough–Tocher macro-element tetrahedral grids (where each base tetrahedron is split into 4 sub-tetrahedra by connecting the barycenter with four vertices), cf. [10]. If splitting further a base tetrahedron into 12 sub-tetrahedra (connecting the barycenter with 4 vertices and 4 face-triangle barycenters), the P_k–P_{k-1} is stable for all $k \geq 2$, cf. [11]. On the uniform tetrahedral grids, i.e., each cube is subdivided into 6 tetrahedra, the continuous P_k with discontinuous P_{k-1} mixed finite element is stable for all $k \geq 6$, cf. [12]. With additional constraints on the finite element spaces, Neilan showed that the P_k–P_{k-1} element is stable for $k \geq 6$ on general tetrahedral

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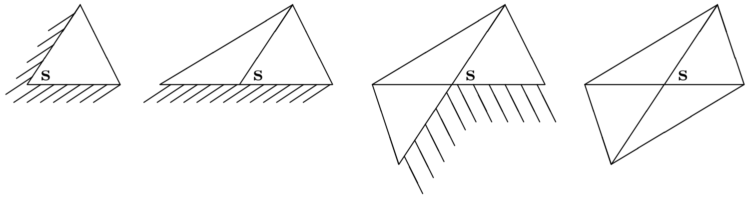

Fig. 2.1. There boundary singular points (left three) and an internal singular point.

grids if the continuous velocity finite element is C_2 continuous at all vertices and also C_1 continuous on all edges, and the discontinuous pressure finite element function is C_1 at all vertices and C_0 on all edges, cf. [13]. But the Scott–Vogelius problem is still open, on general tetrahedral grids. On rectangular grids, this problem is simple that $Q_{k,k-1} \times Q_{k-1,k}$–$Q_{k-1}$ element (and its nD version) is stable for all $k \geq 2$, where $Q_{k,k-1}$ denotes the continuous piecewise polynomials of separated degrees k and $k-1$ in its first variable and second variable, respectively, cf. [14–16].

The mixed finite element of continuous P_3 velocity and discontinuous P_2 pressure is not stable on general triangular grids in 2D, cf. [1,2]. In this work, we enrich the continuous P_3 space by some divergence-free P_4 bubble functions. Such P_4 bubble functions do not provide additional approximation power, but do provide additional degrees of freedom to relax the locking problem of the divergence-free constraint. Such a finite element enrichment technique is used before, many times. For example, mentioned above, Guzman and Neilan enrich the continuous P_3 velocity by some rational bubble functions to obtain an inf–sup stable mixed finite element [7,8]. Here, instead of rational functions (whose numerical integration formula is unknown) we use the P_4 bubble polynomials in this work. In the low order mixed finite element methods for the linear elasticity equation, the $H(\mathrm{div})$ P_k finite element space must be enriched by P_{d+1} divergence-free bubble functions in d-dimensional space, cf. [17–21].

2. The enriched P_3 divergence-free element

In this section, we define the P_4-enriched P_3 divergence-free finite element. Its uni-solvence is shown.

We consider a model stationary Stokes problem: Find the velocity $\mathbf{u}$ and the pressure p on a 2D polygonal domain Ω, such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The standard variational form for (2.1) is: Find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega) := L^2(\Omega)/C = \{p \in L^2 \mid \int_\Omega p = 0\}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in L_0^2(\Omega). \end{aligned} \tag{2.2}$$

Here $H_0^1(\Omega)^2$ is the subspace of the Sobolev space $H^1(\Omega)^2$ (cf. [22]) with zero boundary trace, and the bilinear forms are defined by

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx, \\ b(\mathbf{v}, p) &= -\int_\Omega \operatorname{div} \mathbf{v} \, p \, dx, \\ (\mathbf{f}, \mathbf{v}) &= \int_\Omega \mathbf{f} \, \mathbf{v} \, dx. \end{aligned}$$

Let $\mathcal{T}_h$ be an initial triangulation of Ω. We refine each triangle into four congruent triangles by connecting the three mid-edge points. This way, one grid is refined to the next level grid. We denote each triangulation in the sequence of grids also by $\mathcal{T}_h$, where h is the grid size. Here we introduce the multigrids [23,24], instead of general quasi-uniform grids, to avoid the technical details of nearly-singular points. For an initial triangulation, we may have a few singular points [1,25]. Here a singular point is a point at which all edges of an triangulation fall into two crossing lines at the point. There are exactly four types of singular points, three boundary ones and one internal one, shown in Fig. 2.1. There is a minor constraint for the discrete pressure functions at the singular points. However, all singular points of the initial grid will stay singular and no new singular points appear, after the multigrid refinement.

Let the P_4-enriched P_3 velocity space be, for $K \in \mathcal{T}_h$,

$$\mathbf{V}_K = \{\mathbf{v} \in P_4(K)^2 \mid \operatorname{div} \mathbf{v} \in P_2\}. \tag{2.3}$$

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