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The imaging of small perturbations in an anisotropic media

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Dedicated to Peter Monk on the occasion of his 60th birthday

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ABSTRACT

In this paper, we employ asymptotic analysis to determine information about small volume defects in a known anisotropic scattering medium from far field scattering data. The location of the defects is reconstructed via the MUSIC algorithm from the range of the multi-static response matrix derived from the asymptotic expansion of the far field pattern in the presence of small defects. Since the same data determines the transmission eigenvalues corresponding to the perturbed media, we investigate how the presence of the defects changes the transmission eigenvalues and use this information to recover the strength of the small defects. We provide convergence results on transmission eigenvalues as the size of the defects tends to zero as well as derive the first correction term in the asymptotic expansion of the simple transmission eigenvalues. Numerical examples are presented to show the viability of our imaging method.

1. Introduction

The imaging of anisotropic media from scattering data is a challenging problem mainly due to the non-uniqueness issue [1]. Yet, in many applications in medical imaging and non-destructive testing, the scattering media exhibit anisotropic properties in the interaction with probing waves. The so-called qualitative methods in inverse scattering [2] provide imaging techniques to obtain information on changes in material properties of a known anisotropic media. This work concerns the imaging of small volume (possibly anisotropic) perturbations of a known anisotropic inhomogeneous media in acoustic wave propagation (for the case of $\mathbb{R}^3$) or specially polarized electromagnetic wave propagation (for the case of $\mathbb{R}^2$). Combining asymptotic analysis with MUSIC and the related transmission eigenvalue problem we derive a range test for the location of small perturbations and computable formulas that provide information about the strength (involving the contrast and geometrical features) of the small perturbation. There is a vast literature on the MUSIC algorithm for a variety of scattering problems [3–7] and we recall here its formulation for the anisotropic inhomogeneous media. The asymptotic analysis of the transmission eigenvalue problem for isotropic media is studied in [8] and [9]. One of the main contributions of this study is the asymptotic analysis of the transmission eigenvalue problem for anisotropic media with the first order correction term for the perturbation of the eigenvalues. Note that the transmission eigenvalue problem is non-linear and non-selfadjoint, and the mathematical structure of this problem for anisotropic media is different from the isotropic case. In addition, we show how to use the asymptotic expansion for the perturbation of transmission eigenvalues together with the MUSIC algorithm to image small volume perturbations of anisotropic media.

Let us now precisely formulate the problem under consideration. To this end let $D \subset \mathbb{R}^d$ (for $d = 2$ or 3) be a bounded domain with piecewise smooth boundary which denotes the support of the anisotropic media to be tested. The real valued

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symmetric matrix $A(x) \in C^1(D, \mathbb{R}^{d\times d})$ with smooth entries and the smooth function $n \in C^1(D)$ represent the constitutive parameters for the unperturbed ("healthy") anisotropic media. Without loss of generality we assume that outside the scatterer D the background media has refractive index scaled to one, i.e. $A(x) = I$ and $n(x) = 1$ in $x \in \mathbb{R}^d \setminus \overline{D}$, where I denotes the identity matrix. We define

$$A_b(x) = \begin{cases} I & x \in \mathbb{R}^d \setminus \overline{D} \\ A(x) & x \in D \end{cases} \quad \text{and} \quad n_b(x) = \begin{cases} 1 & x \in \mathbb{R}^d \setminus \overline{D} \\ n(x) & x \in D. \end{cases}$$

Now the scattering of a time harmonic incident plane wave $e^{ikx\cdot\hat{y}}$ with incident direction $\hat{y} \in \mathbb{S}$ by the unperturbed media (i.e. without defects) is mathematically formulated as: find $u_b \in H^1_{loc}(\mathbb{R}^d)$ with $u_b = u_b^s + e^{ikx\cdot\hat{y}}$ such that

$$\nabla \cdot A_b(x)\nabla u_b + k^2 n_b(x) u_b = 0 \quad \text{in } \mathbb{R}^d \tag{1}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_b^s}{\partial r} - iku_b^s \right) = 0, \tag{2}$$

where $\mathbb{S}$ denotes the unit circle/sphere, $r = |x|$, and the Sommerfeld radiation condition (2) is satisfied uniformly with respect to $\hat{x} = x/|x|$. Here u_b is the total field in the background (including the homogeneous part and the media of compact support $\overline{D}$) and u_b^s is the scattered field due to the region D. Recall that the scattered radiating field $u_b^s(\cdot, \hat{y})$, which depends on the incident direction $\hat{y}$, has the following asymptotic expansion [10]

$$u_b^s(x, \hat{y}) = \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} \left\{ u_b^\infty(\hat{x}, \hat{y}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \to \infty \tag{3}$$

where $\hat{x} := x/|x|$, and $u_b^\infty(\hat{x}, \hat{y})$, which depends on the incident direction $\hat{y}$ and observation direction $\hat{x}$, is the corresponding far field pattern. Now we consider the small defective regions that are given by $z_m + \varepsilon B_m$ where B_m is a smooth deformation of a ball centered at the origin. Let A_m and n_m be constant constitutive parameters for the defective regions given by $z_m + \varepsilon B_m$ and assume that

$$|z_i - z_j| \ge c_0 > 0 \quad \text{for all } i \ne j \text{ with } i, j = 1, 2, \ldots, M \quad \text{and}$$
$$\operatorname{dist}(z_m, \partial D) \ge c_0 > 0 \quad \text{for all } m = 1, 2, \ldots, M.$$

The union of the defective regions is denoted by $D_\varepsilon = \bigcup_{m=1}^M (z_m + \varepsilon B_m)$ and we let

$$A_\varepsilon(x) = \begin{cases} A_m & x \in (z_m + \varepsilon B_m) \\ A_b(x) & x \in \mathbb{R}^d \setminus \overline{D}_\varepsilon \end{cases} \quad \text{and} \quad n_\varepsilon(x) = \begin{cases} n_m & x \in (z_m + \varepsilon B_m) \\ n_b(x) & x \in \mathbb{R}^d \setminus \overline{D}_\varepsilon. \end{cases}$$

The scattering problem for the media with the defective region D_ε now reads: find $u_\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ with $u_\varepsilon = u_\varepsilon^s + e^{ikx\cdot\hat{y}}$ such that

$$\nabla \cdot A_\varepsilon(x)\nabla u_\varepsilon + k^2 n_\varepsilon(x) u_\varepsilon = 0 \quad \text{in } \mathbb{R}^d \tag{4}$$

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s \right) = 0. \tag{5}$$

Similarly since u_ε^s is a radiating solution to the Helmholtz equation in $\mathbb{R}^d \setminus \overline{D}$, it assumes a similar asymptotic expansion as (3), and we denote by $u_\varepsilon^\infty(\hat{x}, \hat{y})$ its corresponding far field pattern. In this study we assume that the media is non-absorbing, and $\inf_{x\in D} n(x) = n_0 > 0$, $n_m > 0$, and

$$\inf_{x\in D} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{\min} > 0 \quad \text{and} \quad \sup_{x\in D} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{max} < \infty. \tag{6}$$

For later use let us denote

$$\min_{m=1\ldots M} \inf_{|\xi|=1} \overline{\xi} \cdot A_m\xi = a_{\min} > 0 \quad \text{and} \quad \max_{m=1\ldots M} \sup_{|\xi|=1} \overline{\xi} \cdot A_m\xi = a_{max} < \infty. \tag{7}$$

The *inverse problem* we consider here is to determine the location $\{z_m\}_{m=1,M}$ of the perturbations and information about A_m and n_m from knowledge of $u_\varepsilon^\infty(\hat{x}, \hat{y})$ for several $\hat{x}, \hat{y} \in \mathbb{S}$, provided that $A_b(x)$ and $n_b(x)$ are known.

In general, the support D_ϵ of the defects can be determined from the *far field operator*

$$(Fg)(\hat{x}) = \int_{\mathbb{S}} \left[u_\varepsilon^\infty(\hat{x}, \hat{y}) - u_b^\infty(\hat{x}, \hat{y}) \right] g(\hat{y})\, d\hat{y} \quad \hat{x} \in \mathbb{S} \tag{8}$$

via the factorization method [11]. In addition, it is well-known [2] that the far field operator F determines the *real transmission eigenvalues* which are defined below.