ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A posteriori error estimation for the Laplace–Beltrami equation on spheres with spherical splines

Duong Thanh Pham *, Tung Le

Computational Engineering, Vietnamese German University, Le Lai street, Binh Duong New City, 0274 Binh Duong Province, Viet Nam

ARTICLE INFO

Article history:
Received 13 August 2016
Received in revised form 26 June 2017
Accepted 4 July 2017
Available online xxxx

Keywords:
Laplace–Beltrami equation
Spherical spline
A posteriori error estimate
Adaptivity

ABSTRACT

We prove a posteriori upper and lower bounds for the error estimates when solving the Laplace–Beltrami equation on the unit sphere by using the Galerkin method with spherical splines. Adaptive mesh refinements based on these a posteriori error estimates are used to reduce complexity and computational cost of the corresponding discrete problems. The theoretical results are corroborated by numerical experiments.

1. Introduction

Partial differential equations on the sphere have many applications, for example in weather forecasting models and geophysics; see e.g. [1–3]. Efficient solutions to these equations have become more and more demanding when satellites have been launched into the space to collect data. In this paper, we consider the model equation

$$-\Delta_{\mathbb{S}} u + \omega^2 u = f \quad \text{on } \mathbb{S}, \tag{1.1}$$

where $\Delta_{\mathbb{S}}$ is the Laplace–Beltrami operator, ω is some nonzero real constant and $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$, that is, $\mathbb{S} = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| \times = 1\}$. Here, $|\cdot|$ denotes the Euclidean norm. This elliptic equation arises, for example, when one discretizes in time the diffusion equation on the sphere.

When solving the Laplace–Beltrami equation on the unit sphere, tensor products of univariate splines are not a good choice if data locations are not spaced over a regular grid. Spherical radial basis functions seem to be a better choice [4]; however, the resulting matrix systems from this approximation are very ill-conditioned. Another alternative is to use spherical splines, which are piecewise homogeneous polynomials defined on spherical triangulations.

Sharing many properties in common with classical polynomial splines over planar triangulations, spherical splines [5–7] are well suited for scattered data interpolation and approximation problems. Baramidze and Lai [8] use these functions to solve the Laplace–Beltrami equation on the unit sphere in $\mathbb{R}^3$. Pham et al. later use these functions to solve general pseudodifferential equations on the unit sphere, see [9]. A priori error estimate is proved when solving the equation by using the Galerkin method with spherical splines,

$$\|e\|_{H^1(\mathbb{S})} \leq C h^{s-1} \|u\|_{H^s(\mathbb{S})},$$

where $s \geq 1$ and C is a constant which is independent of the mesh size h and the exact (unknown) solution u. The a priori error estimate reveals the rate of convergence but is of limited use if one requires a numerical estimate of the accuracy. The

* Corresponding author.
E-mail addresses: ptduong01@gmail.com (D.T. Pham), tung.l@vgu.edu.vn (T. Le).

http://dx.doi.org/10.1016/j.camwa.2017.07.003
0898-1221/

Please cite this article in press as: D.T. Pham, T. Le, A posteriori error estimation for the Laplace–Beltrami equation on spheres with spherical splines, Computers and Mathematics with Applications (2017), http://dx.doi.org/10.1016/j.camwa.2017.07.003.

difficulty is that either the exact solution u and its higher derivatives are unknown or bounds for the constant C are not available.

A posteriori error estimates can meanwhile provide numerical estimates of accuracy in terms of the source term and discrete solutions. In this paper, we shall prove a posteriori upper and lower bounds for the errors when solving the Laplace–Beltrami equation on the unit sphere by using Galerkin method with spherical splines. Based on these a posteriori estimates in which approximation errors are bounded (from above and below) by local error estimators, we suggest the use of adaptive mesh refinements to produce better approximate spaces. This results in a significant reduction in required degrees of freedom and computation time while preserving approximate accuracy.

The structure of the paper is as follows. In Section 2, we review spherical splines, introduce the Sobolev spaces on the unit sphere to be used, present the quasi-interpolation operator and the Laplace–Beltrami equation. The proof for an a posteriori upper bound for the error estimate is presented in Section 3 followed by the proof for a lower bound in Section 4. In Section 5, we discuss a simple adaptive mesh refinement algorithm based on these a posteriori error estimates. The final section (Section 6) presents our numerical experiments which illustrate our theoretical results.

In this paper C and C_i, for $i = 1, \dots, 5$, denote generic constants which may take different values at difference occurrences.

2. Preliminaries

In this section, we first review spherical splines [5–7] and introduce our functional spaces on the unit sphere $\mathbb{S} \subset \mathbb{R}^3$. Then the quasi-interpolation operator and the Laplace–Beltrami equation will be discussed.

2.1. Spherical splines

Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be linearly independent vectors in $\mathbb{R}^3$. The *trihedron* T generated by $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is defined by

$$T = \{\boldsymbol{v} \in \mathbb{R}^3 : \boldsymbol{v} = b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + b_3\boldsymbol{v}_3 \text{ with } b_i \geq 0,\ i = 1, 2, 3\}.$$

The intersection $\tau = T \cap \mathbb{S}$ is called a *spherical triangle*. Let $\mathscr{T} = \{\tau_i : i = 1, \dots, \mathcal{T}\}$ be a set of spherical triangles. Then $\mathscr{T}$ is called a *spherical triangulation* of the sphere $\mathbb{S}$ if there hold

1. $\bigcup_{i=1}^{\mathcal{T}} \tau_i = \mathbb{S}$,
2. each pair of distinct triangles in $\mathscr{T}$ is either disjoint or shares a common vertex or an edge.

Let Π_d denote the space of trivariate homogeneous polynomials of degree d in $\mathbb{R}^3$. The space of restrictions on the unit sphere $\mathbb{S}$ of all polynomials in Π_d is denoted by $\Pi_d(\mathbb{S})$. Similarly, we also denote by $\mathcal{P}_d$ and $\mathcal{P}_d(\mathbb{S})$ the spaces of polynomials of degree d in $\mathbb{R}^3$ and on $\mathbb{S}$, respectively. We define $S_d^r(\mathscr{T})$ to be the space of piecewise homogeneous splines of degree d and smoothness r on a spherical triangulation $\mathscr{T}$, that is,

$$S_d^r(\mathscr{T}) = \{s \in C^r(\mathbb{S}) : s|_\tau \in \Pi_d,\ \tau \in \mathscr{T}\}.$$

Throughout this paper, we always assume that

$$\begin{cases} d \geq 3r + 2 & \text{if } r \geq 1 \\ d \geq 1 & \text{if } r = 0 \end{cases}$$

holds; see [5–7].

For a spherical triangle τ with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2$, and $\boldsymbol{v}_3$, let $b_{1,\tau}(\boldsymbol{v})$, $b_{2,\tau}(\boldsymbol{v})$, and $b_{3,\tau}(\boldsymbol{v})$ denote the spherical barycentric coordinates as functions of $\boldsymbol{v}$ in τ, i.e.,

$$\boldsymbol{v} = b_{1,\tau}(\boldsymbol{v})\boldsymbol{v}_1 + b_{2,\tau}(\boldsymbol{v})\boldsymbol{v}_2 + b_{3,\tau}(\boldsymbol{v})\boldsymbol{v}_3.$$

We define the homogeneous Bernstein basis polynomials of degree d relative to τ to be the polynomials

$$B_{ijk}^{d,\tau}(\boldsymbol{v}) = \frac{d!}{i!j!k!} b_{1,\tau}(\boldsymbol{v})^i b_{2,\tau}(\boldsymbol{v})^j b_{3,\tau}(\boldsymbol{v})^k, \quad i + j + k = d.$$

As was shown in [5], we can use these polynomials as a basis for Π_d.

A spherical cap centered at $\boldsymbol{x} \in \mathbb{S}$ and having radius R is defined by

$$C(\boldsymbol{x}, R) = \{\boldsymbol{y} \in \mathbb{S} : \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) \leq R\}.$$

For any spherical triangle τ, let $|\tau|$ denote the diameter of the smallest spherical cap containing τ, and ρ_τ denote the diameter of the largest spherical cap contained in τ. We define

$$|\mathscr{T}| = \max\{|\tau| : \tau \in \mathscr{T}\} \quad \text{and} \quad \rho_{\mathscr{T}} = \min\{\rho_\tau : \tau \in \mathscr{T}\},$$

and refer to $|\mathscr{T}|$ as the mesh size. Our triangulations are said to be *regular* if for some given $\beta > 1$, there holds

$$|\tau| \leq \beta \rho_\tau \quad \forall \tau \in \mathscr{T} \tag{2.1}$$