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Domain decomposition methods for the diffusion equation with low-regularity solution

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ABSTRACT

We analyze matching and non-matching domain decomposition methods for the numerical approximation of the mixed diffusion equations. Special attention is paid to the case where the solution is of low regularity. Such a situation commonly arises in the presence of three or more intersecting material components with different characteristics. The domain decomposition method can be non-matching in the sense that the traces of the finite element spaces may not fit at the interface between subdomains. We prove well-posedness of the discrete problem, that is solvability of the corresponding linear system, provided two algebraic conditions are fulfilled. If moreover the conditions hold independently of the discretization, convergence is ensured.

1. Introduction

The diffusion equation can model different physical phenomena such as Darcy's law, Fick's law or the neutron diffusion. When formulated as a mixed system of equations, it allows to compute both the solution and its gradient. Hence, from a variational point of view, two approaches coexist. One uses either the primal variational formulation to focus on the solution; or the dual-mixed variational formulation to focus instead on the gradient of the solution.

The numerical analysis of domain decomposition methods for the mixed diffusion equation has already been studied for Darcy's law, see e.g. [1–4], and also for Fick's law and the neutron diffusion equation, see e.g. [5–7]. In order to handle non-matching discretizations at the interface of the subdomains, a Lagrange multiplier can be introduced. This technique is known as the mortar finite element method [8], among its predecessors one can cite the hybrid finite element method (see [9] for the diffusion equation). From an algebraic point of view, the linear system obtained after discretization is similar to the one of the Schur complement method.

In this paper, we focus on matching and non-matching domain decompositions of the (dual) mixed formulation. We put special emphasis on the so-called *low-regularity solutions*. For the diffusion equation, we recall that the solution always belongs to the Sobolev space H^1. However, it may happen that the a priori regularity result (even for smooth, locally supported data) only guarantees that the solution is piecewise H^{1+r}, where $r > 0$ can be arbitrarily small: one says that the problem is H^{1+r}-regular. On the other hand, in the above mentioned references, when the regularity issue is explicitly taken into account, it is assumed that the solution is at least:

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- piecewise H^{1+r} with $r > 1/4$ in [7];
- piecewise H^{1+r} with $r > 1/2$ in [1,2,4];
- piecewise H^{1+r} with $r = 1$ in [3,5].

So one aim of our paper is to devise a method that can be fully justified for a problem that is H^{1+r}-regular, with an exponent $r > 0$ that can be arbitrarily small. In other words, we address the "technicalities" needed for the design of a theory that handles low-regularity solutions. To reach that end, one has to modify the existing mathematical and variational frameworks. The other aim is to derive abstract (algebraic) conditions after discretization, to guarantee well-posedness of the discrete problem, and convergence.

The outline of the paper is as follows. In Section 2, we introduce the notations, geometry and Hilbert spaces to define the problem setting. In particular, we will make use of vector-valued functions with L^2-jump of normal traces on the interface between subdomains. Then, in Section 3, we write the continuous equations and the associated variational formulations of the mixed diffusion equations. We also define the *low-regularity case*. We next propose an equivalent multi-domain formulation, which fits into the category of domain decomposition methods. The well-posedness of the mixed, multi-domain formulation is studied in Section 4 in the continuous case and in Section 5 in the discrete case. In the discrete case, we exhibit two abstract algebraic conditions which imply the existence of a discrete inf–sup condition. This inf–sup condition ensures well-posedness of the discrete problem, and also convergence when it is uniform. In addition, these algebraic conditions drive the choice of the space of the Lagrange multipliers. We give numerical illustrations in Section 6. Finally, we draw some conclusions and give perspectives in Section 7.

2. Geometry, Hilbert spaces and notations

Throughout the paper, C is used to denote a generic positive constant which is independent of the meshsize, the triangulation and the quantities/fields of interest. We also use the shorthand notation $A \lesssim B$ for the inequality $A \leq CB$, where A and B are two scalar quantities, and C is a generic constant. Respectively, $A \eqsim B$ for the inequalities $A \lesssim B$ and $B \lesssim A$.

Vector-valued (resp. tensor-valued) function spaces are written in boldface character (resp. blackboard bold characters). Given an open set $O \subset \mathbb{R}^d$, $d = 1, 2, 3$, we use the notation $(\cdot|\cdot)_{0,O}$ (resp. $\|\cdot\|_{0,O}$) for the $L^2(O)$ and $\mathbf{L}^2(O) := (L^2(O))^d$ scalar products (resp. norms). More generally, $(\cdot|\cdot)_{s,O}$ and $\|\cdot\|_{s,O}$ (resp. $|\cdot|_{s,O}$) denote the scalar product and norm (resp. semi-norm) of the Sobolev spaces $H^s(O)$ and $\mathbf{H}^s(O) := (H^s(O))^d$, for $s \in \mathbb{R}$ (resp. for $s > 0$).

If moreover the boundary ∂O is Lipschitz, $\mathbf{n}$ denotes the unit outward normal vector field to ∂O. Finally, it is assumed that the reader is familiar with vector-valued function spaces related to the diffusion equation, such as $\mathbf{H}(\mathrm{div}\,; O)$, $\mathbf{H}_0(\mathrm{div}\,; O)$ etc.

We let $\boldsymbol{\mathcal{R}}$ be a bounded, connected and open subset of $\mathbb{R}^d$, having a Lipschitz boundary which is piecewise smooth. We split $\boldsymbol{\mathcal{R}}$ into N open disjoint parts, or subdomains, $(\boldsymbol{\mathcal{R}}_i)_{i=1,N}$ with Lipschitz, piecewise smooth boundaries: $\overline{\boldsymbol{\mathcal{R}}} = \cup_{i=1,N}\overline{\boldsymbol{\mathcal{R}}_i}$ and the set $\{\boldsymbol{\mathcal{R}}_i\}_{i=1,N}$ is called a partition of $\boldsymbol{\mathcal{R}}$. For a field v defined over $\boldsymbol{\mathcal{R}}$, we shall use the notations $v_i = v_{|\boldsymbol{\mathcal{R}}_i}$, for $i = 1, N$.

Given a partition $\{\boldsymbol{\mathcal{R}}_i\}_{i=1,N}$ of $\boldsymbol{\mathcal{R}}$, let us introduce now function spaces with piecewise regular elements:

$$\begin{aligned}
\boldsymbol{\mathcal{P}}H^s(\boldsymbol{\mathcal{R}}) &= \left\{ \psi \in L^2(\boldsymbol{\mathcal{R}}) \,|\, \psi_i \in H^s(\boldsymbol{\mathcal{R}}_i),\ i = 1, N \right\},\ s > 0;\\
\boldsymbol{\mathcal{P}}\mathbf{H}(\mathrm{div}\,, \boldsymbol{\mathcal{R}}) &= \left\{ \mathbf{q} \in \mathbf{L}^2(\boldsymbol{\mathcal{R}}) \,|\, \mathbf{q}_i \in \mathbf{H}(\mathrm{div}\,, \boldsymbol{\mathcal{R}}_i),\ i = 1, N \right\};\\
\boldsymbol{\mathcal{P}}W^{1,\infty}(\boldsymbol{\mathcal{R}}) &= \left\{ \psi \in L^\infty(\boldsymbol{\mathcal{R}}) \,|\, \psi_i \in W^{1,\infty}(\boldsymbol{\mathcal{R}}_i),\ i = 1, N \right\}.
\end{aligned}$$

Given a partition $\{\boldsymbol{\mathcal{R}}_i\}_{i=1,N}$ of $\boldsymbol{\mathcal{R}}$, we denote by Γ_{ij} the interface between two subdomains $\boldsymbol{\mathcal{R}}_i$ and $\boldsymbol{\mathcal{R}}_j$, for $i \neq j$: if the Hausdorff dimension of $\overline{\boldsymbol{\mathcal{R}}_i} \cap \overline{\boldsymbol{\mathcal{R}}_j}$ is $d-1$, then $\Gamma_{ij} = int(\overline{\boldsymbol{\mathcal{R}}_i} \cap \overline{\boldsymbol{\mathcal{R}}_j})$; otherwise, $\Gamma_{ij} = \emptyset$. By construction, $\Gamma_{ij} = \Gamma_{ji}$. We define the interface Γ_S, respectively the wirebasket $\partial\Gamma_W$ by

$$\Gamma_S = \bigcup_{i=1}^{N}\bigcup_{j=i+1}^{N} \overline{\Gamma_{ij}}, \quad \partial\Gamma_W = \bigcup_{i=1}^{N}\bigcup_{j=i+1}^{N} \partial\Gamma_{ij}.$$

When $d = 2$, the wirebasket consists of isolated crosspoints. When $d = 3$, the wirebasket consists of open edges and crosspoints. Introduce the subset of indices $\mathcal{I}_S := \{i \in \{1, \ldots, N\} \,|\, \partial\boldsymbol{\mathcal{R}}_i \cap \Gamma_S = \partial\boldsymbol{\mathcal{R}}_i\}$ and, for $i = 1, N$, the open set $\Gamma_i = \partial\boldsymbol{\mathcal{R}}_i \setminus \overline{\Gamma_S}$. Let us define function spaces with zero boundary condition, for $i = 1, N$:

$$\begin{aligned}
H^1_{0,\Gamma_i}(\boldsymbol{\mathcal{R}}_i) &= \left\{ \psi \in H^1(\boldsymbol{\mathcal{R}}_i) \,|\, \psi_{|\Gamma_i} = 0 \right\};\\
\boldsymbol{\mathcal{P}}H^1_0(\boldsymbol{\mathcal{R}}) &= \left\{ \psi \in L^2(\boldsymbol{\mathcal{R}}) \,|\, \psi_i \in H^1_{0,\Gamma_i}(\boldsymbol{\mathcal{R}}_i),\ i = 1, N \right\}.
\end{aligned}$$

When $\Gamma_{ij} \neq \emptyset$, let $H^{1/2}_{\Gamma_{ij}}$ be the set of $H^{1/2}(\Gamma_{ij})$ functions whose continuation by 0 to $\partial\boldsymbol{\mathcal{R}}_i$ belongs to $H^{1/2}(\partial\boldsymbol{\mathcal{R}}_i)$. One can prove that $H^{1/2}_{\Gamma_{ij}} = H^{1/2}_{\Gamma_{ji}}$.

For $\mathbf{p} \in \boldsymbol{\mathcal{P}}\mathbf{H}(\mathrm{div}\,, \boldsymbol{\mathcal{R}})$, let us set $[\mathbf{p}\cdot\mathbf{n}]_{ij} := \sum_{k=i,j} \mathbf{p}_k \cdot \mathbf{n}_{k|\Gamma_{ij}}$ the jump of the normal component of $\mathbf{p}$ on Γ_{ij}, when $\Gamma_{ij} \neq \emptyset$. $[\mathbf{p}\cdot\mathbf{n}]_{ij}$ is well defined in $(H^{1/2}_{\Gamma_{ij}})'$ the dual space of $H^{1/2}_{\Gamma_{ij}}$ (see e.g. [10]). The *global jump* $[\mathbf{p}\cdot\mathbf{n}]$ of the normal component on the interface is defined by:

$$[\mathbf{p}\cdot\mathbf{n}]_{|\Gamma_{ij}} := [\mathbf{p}\cdot\mathbf{n}]_{ij}, \ \text{for } i, j = 1, N,\ i < j.$$

By definition, it holds $[\mathbf{p}\cdot\mathbf{n}] \in \prod_{i<j}(H^{1/2}_{\Gamma_{ij}})'$.

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