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Analysis on inexact block diagonal preconditioners for elliptic PDE-constrained optimization problems[☆]

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A B S T R A C T

By using the Galerkin finite element method, the distributed control problems can be discretized into a saddle point problem with a coefficient matrix of block three-by-three, which can be reduced to a linear system with lower order. We first introduce a class of inexact block diagonal preconditioners and estimate the lower and upper bounds of positive and negative eigenvalues of the preconditioned matrices, respectively. Based on the Cholesky decomposition of the known matrices, we also analyze a lower triangular preconditioner to accelerate the minimal residual method for the reduced linear system and discuss its real and complex eigenvalues respectively. Moreover, these bounds do not rely on the regularization parameter and the eigenvalues of the matrices in the discrete system. Numerical experiments are also presented to demonstrate the effectiveness and robustness of the two new preconditioners.

1. Introduction

In this work, we shall mainly consider the efficient preconditioning techniques to the following linear elliptic distributed control problems:

$$\begin{aligned} \min_{u,f} \quad & \frac{1}{2}\|u-\hat{u}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|f\|^2_{L^2(\Omega)}, \\ \text{s.t.} \quad & -\Delta u = f \quad \text{in } \Omega, \\ & \quad u = g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with boundary $\partial\Omega$, u is the state, $\hat{u}$ is a desired state, f is the control, g is given Dirichlet boundary data and β is a regularization (or Tikhonov) parameter. The problem (1.1) was originally introduced by Lions in [1]. Nowadays there are already many investigations available in literature for solving the distributed control problems of form (1.1); see [2–9]. Some other related problems, which include control constraints or state constraints, have also been studied in [2,3,10–12].

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By using the discretize-then-optimize approach, the PDE-constrained optimization problem (1.1) can be transformed into a linear system of the saddle point form, see, e.g., [8]. More precisely, by employing the Galerkin finite element method to Poisson equation in (1.1), we can derive the finite dimensional discrete analogue of the minimization problem [8,9]

$$\begin{aligned} \min_{u,f} \quad & \frac{1}{2}u^T M u - u^T b + \alpha + \frac{\beta}{2} f^T M f, \\ \text{s.t.} \quad & Ku = Mf + d, \end{aligned} \tag{1.2}$$

where $M \in \mathbb{R}^{n\times n}$ is the mass matrix, $K \in \mathbb{R}^{n\times n}$ is the stiffness matrix (the discrete Laplacian), $\alpha = \frac{1}{2}\|\hat{u}\|^2_{L^2(\Omega)}$, $b \in \mathbb{R}^n$ is the Galerkin projection of the target function $\hat{u}$, and $d \in \mathbb{R}^n$ contains the terms arising from the boundary values of the discrete solution. We should emphasize that both M and K are symmetric positive definite, so the minimization problem (1.2) is a convex optimization problem. By applying the Lagrange multiplier technique to the problem (1.2), it follows the saddle point linear systems:

$$\begin{pmatrix} \beta M & 0 & -M \\ 0 & M & K \\ -M & K & 0 \end{pmatrix} \begin{pmatrix} f \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ d \end{pmatrix}, \tag{1.3}$$

where v is a vector of Lagrange multipliers, see, e.g., [13]. On account of the large and sparse of the matrices M and K, iterative methods should be more effective and efficient than direct methods. Although such linear system of form (1.3) can be seen as a special case of the standard saddle point problem:

$$\begin{pmatrix} H & D^T \\ D & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} g \\ d \end{pmatrix},$$

which has been extensively investigated in recent years, see, [14–23,30] and the relative references therein, constructing efficient solvers based on the special structures of the coefficient matrix in (1.3) is still necessary. Recently, some researchers have devoted themselves to construct and study reliable preconditioners for solving the saddle point systems of form (1.3); see, [6–9]. In [8], the authors studied the MINRES method incorporated with a block diagonal preconditioner and the projected preconditioned conjugate gradient method coupled with a constraint preconditioner to derive the numerical solution of the system (1.3) efficiently. Besides, a block counter diagonal preconditioner and a block counter tridiagonal preconditioner have been proposed in [9] to precondition the Krylov subspace methods such as GMRES. The authors in [24] designed a class of block diagonal and block triangular preconditioners with appropriate approximation blocks to solve the system (1.3).

From the first equation in (1.3), we can see that $f = \frac{v}{\beta}$. This implies that the $3n \times 3n$ linear system of form (1.3) can be reduced to

$$\mathcal{A}z =: \begin{pmatrix} M & K \\ K & -\frac{1}{\beta}M \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} =: \ell, \tag{1.4}$$

whose size is $2n \times 2n$. In [25], the authors proposed several kinds of preconditioners to solve the system (1.4). As the size of the system (1.4) is smaller than that of (1.3), in this paper, we first introduce a class of inexact block diagonal preconditioners by using the preconditioners of M and the Schur complement matrix $S =: \frac{1}{\beta}M + KM^{-1}K$ and estimate the lower and upper bounds of positive and negative eigenvalues of the preconditioned matrices, respectively. Particularly, for the exact block diagonal preconditioner, all eigenvalues of the preconditioned matrix can be explicitly expressed in terms of the eigenvalues of $M^{-1}K$ after some detailed analysis by taking full advantage of its special structure and the eigenvalue decomposition of $M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$. Moreover, all these eigenvalues are contained in the interval $(-1, \frac{1-\sqrt{5}}{2}) \cup (1, \frac{1+\sqrt{5}}{2})$, which evidently does not rely on the parameter β and the eigenvalues of $M^{-1}K$. In addition, based on the Cholesky decomposition of M, K and S, we will also study a lower triangular preconditioner with bilateral preconditioning and derive the lower and upper bounds of positive and negative eigenvalues of the preconditioned system. Under some suitable conditions, these bounds will be independent of the parameter β and the eigenvalues of $M^{-1}K$ as well.

The organization of this paper is as follows. We present a class of inexact block diagonal preconditioner and derive all the eigenvalues of the preconditioned matrix in Section 2. Then we study another lower triangular preconditioner and give some explicit and sharp estimates for the spectral bounds of the preconditioned system in Section 3. Numerical experiments are presented in Section 4 to show the effectiveness of our methods.

We end this section with an introduction of some notation that will be used in the subsequent analysis. For $H \in \mathbb{R}^{n\times n}$, we shall often write H^{-1}, H^T and $\|H\|$ to denote the inverse, the transpose and the norm of H, respectively. $H_1 \sim H_2$ will be used for describing that H_1 is similar to H_2. In addition, we use $\|x\|$ to denote the norm of any vector $x \in \mathbb{R}^n$, and I a general identity matrix.

2. Block diagonal preconditioner

As it is known, the convergence rate of the Krylov subspace methods, such as MINRES and GMRES, is closely related to the eigenvalues and the eigenvectors of the coefficient matrix in the concerned linear system [26–29]. But the eigenvalues