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A regularity criterion for a generalized Hall-MHD system

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ABSTRACT

This paper proves a regularity criterion for a 3D generalized Hall-MHD system in terms of velocity gradient in negative Besov spaces.

1. Introduction

In this paper, we consider the following 3D generalized Hall-MHD system:

$$\partial_t u + u\cdot\nabla u + \nabla\left(\pi + \frac{1}{2}|b|^2\right) + (-\Delta)^\alpha u = b\cdot\nabla b, \tag{1.1}$$

$$\partial_t b + u\cdot\nabla b - b\cdot\nabla u + (-\Delta)^\beta b + \operatorname{rot}(\operatorname{rot} b\times b) = 0, \tag{1.2}$$

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.3}$$

$$(u, b)(\cdot, 0) = (u_0, b_0) \quad in\ \mathbb{R}^3. \tag{1.4}$$

Here u, π and b denote the velocity, pressure and magnetic field of the fluid, respectively. $0 < \alpha, \beta$ are two constants. The fractional Laplacian operator $(-\Delta)^\alpha$ is defined through the Fourier transform, namely, $\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\hat{f}(\xi)$.

The applications of the Hall-MHD system cover a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamo.

When the Hall effect term $\operatorname{rot}(\operatorname{rot} b\times b)$ is neglected, the system (1.1)–(1.4) reduces to the well-known generalized MHD system, which has received many studies [1–4].

When $\alpha = \beta = 1$, the system (1.1)–(1.4) reduces to the well-known Hall-MHD system. The paper [5] gave a derivation of the isentropic Hall-MHD system from a two-fluid Euler–Maxwell system. Chae–Degond–Liu [6] proved the local existence of smooth solutions. Chae and Schonbek [7] showed the time-decay and some regularity criteria were proved in [8,9].

Local well-posedness is established in [10] when $0 < \alpha \le 1$ and $\frac{1}{2} < \beta \le 1$ even for u_0 and b_0 in different spaces.

When $\frac{3}{4} \le \alpha < \frac{5}{4}$ and $1 \le \beta < \frac{7}{4}$, Jiang and Zhu [11] prove the following regularity criteria

$$\nabla b \in L^t(0, T; L^s) \quad \text{with } \frac{2\beta}{t} + \frac{3}{s} \le 2\beta - 1, \qquad \frac{3}{2\beta - 1} < s \le \infty, \tag{1.5}$$

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and one of the following two conditions

$$u \in L^p(0,T;L^q) \quad \text{with } \frac{2\alpha}{p}+\frac{3}{q} \le 2\alpha-1, \qquad \frac{3}{2\alpha-1} < q \le \frac{6\alpha}{2\alpha-1}, \tag{1.6}$$

or

$$\Lambda^\alpha u \in L^p(0,T;L^q) \quad \text{with } \frac{2\alpha}{p}+\frac{3}{q} \le 3\alpha-1, \qquad \frac{3}{3\alpha-1} < q \le \frac{6\alpha}{3\alpha-1}. \tag{1.7}$$

When $1 \le \alpha < \frac{5}{4}$ and $1 \le \beta < \frac{7}{4}$, Ye [12] showed the following regularity criterion

$$u \in L^p(0,T;L^q) \quad \text{and} \quad \nabla b \in L^\ell(0,T;L^k),$$

where p, q, ℓ and k satisfy the relation

$$\frac{2\alpha}{p}+\frac{3}{q} \le 2\alpha-1, \qquad \frac{2\beta}{p}+\frac{3}{q} \le 2\beta-1, \qquad \frac{2\beta}{\ell}+\frac{3}{k} \le 2\beta-1, \tag{1.8}$$

and

$$\max\left(\frac{3}{2\alpha-1},\frac{3}{2\beta-1}\right) < q \le \infty, \qquad \frac{3}{2\beta-1} < k \le \infty.$$

For other related results, we refer to [13–15] and references therein.

Very recently, He–Ahmad–Hayat–Zhou [16] show the following regularity criterion

$$\operatorname{rot} u \in L^{\frac{2\alpha}{2\alpha-2-s_1}}(0,T;\dot{B}^{-s_1}_{\infty,\infty}) \quad \text{with } 0 < s_1 < \min(1, 2\alpha-2), \tag{1.9}$$

when $1 < \alpha = \beta < \frac{7}{4}$, and the following regularity criterion

$$(-\Delta)^{\frac{m-1}{2}} \operatorname{rot} u \in L^2(0,T;L^2) \quad \text{with } m > \frac{7}{2}, \tag{1.10}$$

as $\alpha=\beta=1$. As far as we know, it is the first work to establish regularity criteria only on the velocity field u.

The aim of this paper is to prove a new regularity criterion, we will prove the following theorem.

Theorem 1.1. *Let $0 < \alpha < \frac{5}{4}$ and $\frac{7}{4} \le \beta < 3$ and $u_0, b_0 \in H^s$ with $s \ge 2$ and $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ in $\mathbb{R}^3$. If ∇u satisfy*

$$\nabla u \in L^{\frac{2\alpha}{2\alpha-\gamma}}(0,T;\dot{B}^{-\gamma}_{\infty,\infty}), \qquad \text{with } 0 < \gamma < 2\alpha, \tag{1.11}$$

then the solution (u,b) can be extended beyond T.

In the following proofs, we will use the following bilinear product and commutator estimates due to Kato–Ponce [17]:

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}), \tag{1.12}$$

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{1.13}$$

with $s > 0$, $\Lambda := (-\Delta)^{\frac{1}{2}}$ and $\frac{1}{p} = \frac{1}{p_1}+\frac{1}{q_1} = \frac{1}{p_2}+\frac{1}{q_2}$.

We will also use the improved Gagliardo–Nirenberg inequalities [18–20]:

$$\|\nabla u\|_{L^3}^3 \le C\|\nabla u\|_{\dot{B}^{-\gamma}_{\infty,\infty}}\|\nabla u\|^2_{\dot{H}^{\frac{\gamma}{2}}}, \tag{1.14}$$

$$\|\nabla u\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \le C\|\nabla u\|_{\dot{B}^{-\gamma}_{\infty,\infty}}^{\frac{k+\alpha-1}{k+\gamma+\alpha-1}}\|\nabla u\|_{\dot{H}^{k+\alpha-1}}^{\frac{\gamma}{k+\gamma+\alpha-1}}, \tag{1.15}$$

$$\|\Lambda^k u\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2k+\gamma+2\alpha-2}}} \le C\|\nabla u\|_{\dot{B}^{-\gamma}_{\infty,\infty}}^{\frac{\gamma}{2(k+\gamma+\alpha-1)}}\|\Lambda^k u\|_{\dot{H}^{\frac{\gamma(k+\gamma-1)}{2k+\gamma+2\alpha-2}}}^{\frac{2k+\gamma+2\alpha-2}{2(k+\gamma+\alpha-1)}}, \tag{1.16}$$

for $k > 0$, $0 < \gamma < \alpha$. Some more efficient inequalities was used in [21–23].

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, we only need to prove a priori estimates.

First, testing (1.1) by u and using (1.3), we see that

$$\frac{1}{2}\frac{d}{dt}\int |u|^2 dx + \int |\Lambda^\alpha u|^2 dx = \int (b\cdot\nabla)b\cdot u dx.$$

Testing (1.2) by b and using (1.3), we find that

$$\frac{1}{2}\frac{d}{dt}\int |b|^2 dx + \int |\Lambda^\beta b|^2 dx = \int (b\cdot\nabla)u\cdot b dx.$$