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Blow-up phenomena for a nonlinear reaction–diffusion system with time dependent coefficients

Xueyan Tao [a], Zhong Bo Fang [b,*]

[a] School of Mathematical Sciences, Peking University, Beijing 100871, PR China
[b] School of Mathematical Sciences, Ocean University of China, Qingdao 266100, PR China

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ABSTRACT

We investigate the blow-up phenomena for the solution to a nonlinear reaction–diffusion system with time dependent coefficients subject to null Dirichlet boundary conditions. By virtue of Kaplan's method, method of subsolutions and supersolutions and modified differential inequality technique, we establish the blow-up criteria for the solution. Moreover, lower and upper bounds for the blow-up time are derived.

1. Introduction

In this paper, we consider the reaction–diffusion system with time dependent coefficients

$$\begin{cases} u_t = \Delta u + k_1(t)\, u^p v^q, & (x,t) \in \Omega \times \left(0, t^*\right), \\ v_t = \Delta v + k_2(t)\, v^r u^s, & (x,t) \in \Omega \times \left(0, t^*\right), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times \left(0, t^*\right), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ is a bounded region with smooth boundary $\partial\Omega$, $k_1(t)$, $k_2(t)$ are bounded positive C^1-functions, $p, q, r, s \geq 0$, t^* is a possible blow-up time when blow-up occurs, otherwise $t^* = +\infty$. The nonnegative initial data $u_0(x)$, $v_0(x)$ are C^1-functions which satisfy compatibility conditions. Therefore, by the classical parabolic theory, it follows that the solution to problem (1.1) exists uniquely, and is nonnegative. More precise conditions for other data will be given later.

Reaction system (1.1) models such as heat propagations in a two-component combustible mixture [1]; chemical processes [2]; interaction of two biological groups without self limiting [3], etc.

During the past decades, there have been many works to deal with the blow-up phenomena for the solutions to nonlinear parabolic equations and systems, we refer the reader to the monograph [4] as well as to the survey paper [5] and the references therein. As we all know, solutions to these problems may remain bounded or blow up in finite or infinite time. When blow-up occurs, the derivation for t^* is a problem of great practical interest. While the exact value of t^* is usually difficult to be derived, we hope to obtain the bounds for it. In this paper, we are particularly interested in the estimate for the bounds for the blow-up time of the blow-up solution to a nonlinear reaction–diffusion system. There are many

* Corresponding author.
E-mail addresses: taoxueyankitty@163.com (X. Tao), fangzb7777@hotmail.com (Z.B. Fang).

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results about bounds for the blow-up time to a single reaction equation, we refer the readers to the literature [6,7] (constant coefficients), [8–10] (space dependent coefficients), and [11–14] (time dependent coefficients).

However, few works have devoted to the estimate for the blow-up time of blow-up solutions to the reaction systems. In [15], Payne and Song considered the model of chemotaxis

$$\begin{cases} u_t = \Delta u - a_1 (uv_{,i})_{,i}, & (x,t) \in \Omega \times (0,t^*), \\ v_t = a_2 \Delta v - a_3 v + a_4 u, & (x,t) \in \Omega \times (0,t^*), \end{cases}$$

under homogeneous Neumann boundary conditions, they obtained the lower bounds for the blow-up time of the blow-up solution to the initial boundary value problem in two or three-dimensional spaces. Recently, Xu and Ye [16] investigated the following weak coupled reaction–diffusion problem for large initial data and suitable parameters

$$\begin{cases} u_t = \Delta u + u^p v^q, & (x,t) \in \Omega \times (0,t^*), \\ v_t = \Delta v + v^r u^s, & (x,t) \in \Omega \times (0,t^*), \end{cases}$$

they derived the exact value of the blow-up time under homogeneous Dirichlet boundary conditions. Payne and Philippin [17] considered the semilinear parabolic system with time dependent coefficients as follows

$$\begin{cases} u_t = \Delta u + k_1(t) f_1(v), & (x,t) \in \Omega \times (0,t^*), \\ v_t = \Delta v + k_2(t) f_2(u), & (x,t) \in \Omega \times (0,t^*), \end{cases}$$

under homogeneous Dirichlet boundary conditions, they obtained suffcient conditions for the solution blows up in finite time and therefore derived the upper bounds for the blow-up time. Moreover, they obtained the lower bounds for the blow-up time in two or three-dimensional spaces. However, they did not consider the high-dimensional space case, and in their work, the behavior of time dependent coefficients have no effect on the lower bounds for the blow-up time. Moreover, our model (1.1) is quite different from their models. Besides, we refer [18,19] to see studies about nonlocal reaction systems.

In view of the works mentioned above, there are few results about bounds for the blow-up time of blow-up solution to the initial boundary value problem (1.1). The main difficulties are to seek the competitive relationship between diffusion term and nonlinear source term, as well as to investigate the influence of space dimension and time dependent coefficients to the blow-up solution. Motivated by these observations, using Kaplan's method, method of subsolutions and supersolutions as well as a modified differential inequality technique, we investigate the blow-up criteria for the solution to problem (1.1), and obtain the bounds for the blow-up time in high-dimensional space. Indeed, for the special case $k_1(t) = k_2(t) = 1$, Zheng [20] obtained sufficient conditions for the existence of global and non-global solutions. However, for the completeness of our paper, in Section 2, we give the blow-up criteria and upper bounds for the blow-up time of the solution to problem (1.1) in Kaplan's measure. In Section 3, for high-dimensional space, we obtain lower bounds for the blow-up time of the blow-up solution to problem (1.1) in two different positive measure.

Note that, if we consider the parabolic system (1.1) in a ball or in the whole space, then a large class of radially symmetric decreasing solutions may blows up on a single point [cf. [21]]. To avoid this case, throughout this paper, we assume that the measure of the blow-up set always positive.

2. Criteria for blow-up

In this section, we will combine Kaplan's method with the comparison principle to seek whether the criteria for the solution to problem (1.1) exists globally or blows up in finite time, and therefore derive the upper bounds for the blow-up time.

We consider the fixed membrane problem

$$\Delta\phi + \lambda\phi = 0, \quad x \in \Omega,$$
$$\phi(x) = 0, \quad x \in \partial\Omega,$$

where λ_1, ϕ_1 and μ_1, ψ_1 are the first eigenvalue and the corresponding eigenfunctions for region Ω and $\Omega_\varepsilon := \left\{ x \in \Omega | \, dist\, (x, \partial\Omega) \geq \varepsilon \right\}$ respectively.

Theorem 2.1. *Suppose that k_1, k_2 are bounded functions and let $\underline{K} = \min\{k_1(t), k_2(t)\}$, $\bar{K} = \max\{k_1(t), k_2(t)\}$, and (u, v) is the nonnegative classical solution of problem (1.1).*

(1) *For $p + q \leq 1$ and $r + s \leq 1$, if the initial data are small enough and satisfy (2.6), then the solution of problem (1.1) exists globally.*

(2) *For $p > 1$, if $r + s \geq 1$, the initial data are small enough and satisfy (2.9), then the solution of problem (1.1) exists globally; while if the initial data are large enough and satisfy (2.16), then the solution of problem (1.1) blows up in finite time t^* with the following upper bound*

$$\frac{1}{(1-p)(\mu_1 - k)} \ln \left[1 - \frac{(\mu_1 - k) U^{1-p}(0)}{\underline{K} \delta^q} \right],$$

where $k = -\frac{\lambda_1}{p-1} q$, $U(t) = \int_{\Omega_\varepsilon} \omega \psi_1 dx$, and ω is defined in (2.12).

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