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An adaptive fourth-order partial differential equation for image denoising

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ABSTRACT

To overcome the weakness of second order methods such as Perona–Malik model for image denoising, various high order models have been proposed and studied. However, there is not too much analysis of these equations to be found in the literature. In this paper, we propose an adaptive fourth-order partial differential equation, which joints a fourth-order term and a second-order term. The model takes advantage of the fourth-order model's better image avoiding staircase effect and the second-order model's better edge preserving effect. By introducing a functional framework and k-bounded partial variation (BPV^k) space, we prove the existence of a weak solution of the proposed model. Experimental results show that the proposed model can alleviate the staircase effect and preserve edges accurately.

1. Introduction

Image denoising is a fundamental problem in image processing and computer vision. During the last two decades, researchers have been able to achieve a good dealing with the trade-off between noise removal and edge preservation using the variational and partial differential equations(PDEs) based methods [1], such as anisotropic diffusion second-order PDE models [2–4], total variation TV models [5–7], anisotropic diffusion fourth-order PDE models [8–12] and fractional-order PDE models [13].

Second-order PDE uses decreasing function regard to gradient operator absolute value as integrand of energy functional. Although this type PDE has been demonstrated to be able to achieve a good trade-off between noise removal an edge preservation, it tends to cause the processed image to exhibit staircase effect. In order to reduce the staircase effect, high order PDEs, especially the fourth-order PDE-based methods were proposed to solve this problem in view of their stronger smoothing ability. For example, You and Kaveh [8] proposed the following fourth-order PDE (YK for short) associated with a second energy functional

$$\frac{\partial u}{\partial t} + \Delta(g_1(\Delta u)\Delta u) = 0 \tag{1.1}$$

for noise removal. Diffusion coefficient $g_1(s)$ is a nonnegative monotonically decreasing function. Lihua Min et al. studied in [11] the following fourth-order model (MY for short):

$$\frac{\partial u}{\partial t} + \Delta(K^n |u|^{-n} \Delta u) = 0 \tag{1.2}$$

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where $K > 0$ is the contrast parameter, $n > 0$. Numerical experiments, for example in [8,11], showed that fourth-order models are able to avoid the staircase effect.

However, the fourth-order PDEs suffer from the blurring effect near sharp edges due to the possible over-smoothing. Moreover, in the mathematical analysis, these fourth-order nonlinear PDEs is much less-developed.

In order to tackle the staircase effect and the edge blurring problems, many authors [14–16] proposed some hybrid high order regularization models. In this paper, we consider PDE-based method to solve the two problems.

Our contributions are fourfold:

- We propose an adaptive fourth-order PDE which joints a fourth-order term and a second-order term. In homogeneous regions, we make use of the fourth-order term to reduce the noise and avoid the staircase effect; and near edges, we utilize the second-order term to preserve them.
- We introduce the concept of weighted partial k-order variation, which enables us to define and employ the Banach space of functions of k-bounded partial variation.
- In the theoretical analysis, we prove the existence of a weak solution of the proposed equation, which is very important for the numerical computation.
- We adopt the discrete Fourier transform to implement the numerical algorithm. The experimental results with the comparison of several PDEs models show that the proposed equation is better performance in preserving edges and avoiding the staircase effect. Compared with regularization model, our proposed model is comparable with regularization model [16].

The remainder of this paper is organized as follows. Section 2 details the proposed model. In Section 3, we investigate the existence of a weak solution of the proposed model. Numerical scheme is given in Section 4. In Section 5, we provide numerical experiments to prove the effectiveness of the proposed model in preserving edges as well as avoiding the staircase effect while removing the noise. Finally, Section 6 concludes this paper.

2. The proposed model

We know that the PDE models of Perona–Malik type have strong connections to variational energy minimization problems and this fact is exploited by many to design various diffusion functions [1]. Following [5], consider the next minimization problem for image restoration:

$$\min_u E(u) = \int_\Omega |\nabla u| dxdy + \frac{\lambda}{2}\int_\Omega (u_0 - u)^2 dxdy. \tag{2.1}$$

Here $u_0 : \Omega \to R$ is the observed (noise) image, $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary, and $\lambda > 0$ is a parameter. The formal gradient flow associated with the functional $E(u)$ is given by

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \lambda(u - u_0). \tag{2.2}$$

Motivated by the correspondence between the variational problem (2.1) and PDE (2.2), Surya Prasath et al. [17] proposed adaptive forward-backward diffusion equation (AFBD for short) of the following form

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g(|\nabla G_\sigma * u|)\frac{\nabla u}{|\nabla u|}\right), \tag{2.3}$$

which is gotten from the following minimization problem:

$$\min_u E(u) = \int_\Omega g(|\nabla G_\sigma * u|)|\nabla u| dxdy. \tag{2.4}$$

Here, $g(|\nabla G_\sigma * u|)$ is an edge detector and given by:

$$g(|\nabla G_\sigma * u|) = \frac{1}{1 + K|\nabla G_\sigma * u|^2}$$

where K is a positive constant. G_σ denotes the two-dimensional Gaussian kernel $G_\sigma = (2\pi\sigma)^{-1}\exp(-(x^2+y^2)/2\sigma^2)$ and $*$ denotes the convolution operation. The authors proved the existence of weak solutions for (2.3). Their experimental results showed that their model overcome well-known edge smearing effects of the heat equation by using gradient dependent diffusion function. However, the resulting image in the presence of the noise also show staircase effect. Recently, Dong and Chen [16] proposed a variational model, which used a combination of two different fractional-order derivatives in the regularization. Here, a special form of this model with the integer-order derivative (DC for short) is given by

$$\min_u E(u) = \int_\Omega h_1(x,y)|D^2 u| dxdy + \mu\int_\Omega (1 - h_1(x,y))|\nabla u| dxdy + \frac{\lambda}{2}\int_\Omega (u - u_0)^2 dxdy, \tag{2.5}$$

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