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Steady flow of generalized Newtonian fluid with multivalued rheology and nonmonotone friction law[☆]

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ABSTRACT

We study the stationary incompressible flow of a generalized Newtonian fluid described by a nonlinear multivalued maximal monotone constitutive law and a multivalued nonmonotone frictional boundary condition. We provide results on the existence and uniqueness of a solution to the variational form of the problem. When the multivalued laws are of a subdifferential form, we prove the existence of a solution to a variational-hemivariational inequality for the flow's velocity field.

1. Introduction

In this paper we study a mathematical model that describes the steady flow of incompressible generalized Newtonian fluids in a bounded domain. The fluid is assumed to satisfy a multivalued rheological law between the symmetric part of the velocity gradient and the extra stress tensor. On two components of the boundary of the domain, we assume different boundary conditions: the Dirichlet condition on the velocity on a part where the fluid is supposed to adhere to the boundary, and the multivalued friction condition which holds between the tangential components of the stress tensor and the velocity on the second part.

The multivalued constitutive law considered in the model is given by an inhomogeneous (depending on the spatial variable) maximal monotone graph. Examples of such laws can be found in the modeling of various fluid behaviors in hemodynamics, glaciology, food rheology and polymer industry. In particular, our model includes the so-called Bingham and Herschel–Bulkley fluids, as well as the rigid perfectly plastic fluids (Section 6). The multivalued friction law is given by a closed multifunction with a suitable growth condition. The basic examples of such multifunctions are potential laws described by the Clarke subdifferential of a nonsmooth, nondifferentiable locally Lipschitz function.

Mathematical problems concerning the flow of various kinds of fluids described by the Navier–Stokes equations and their generalizations have been studied in many papers, cf. e.g. [1–9]. Generalized Newtonian fluids which belong to the class of non-Newtonian fluids were treated in Málek et al. [7]. Multivalued power-law rheologies for incompressible flows were considered in [1,10]. The results on stationary non-Newtonian fluids governed by a nonlinear single-valued constitutive law

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and a multivalued nonmonotone subdifferential frictional boundary condition can be found in [11,12]. The frictional contact boundary conditions for fluid have been studied, for both convex and nonconvex potentials, for instance, by Duvaut and Lions [3], Fujita [4], Consiglieri [2], Migórski [13], Migórski and Ochal [14] for the steady flows, and by Fang and Han [15], Kalita and Łukaszewicz [5], Łukaszewicz [6], and Migórski and Ochal [16] for the evolutionary problems.

In this paper, we extend the earlier results of [1,11] in two directions: we establish the existence result for a more general class of admissible constitutive relations and for nonmonotone multivalued boundary condition. Similarly as in [1,17], we assume that the constitutive law is given by a maximal monotone graph. In contrast to [11], the constitutive relation is multivalued which in a particular case of the subdifferential of a convex potential leads to a variational term in the inequality problem for the velocity field. Also, we assume that the classical adherence of the fluid to the boundary enclosing its flow holds only on a part of the boundary and our interest is in a friction nonsmooth multivalued condition. Our approach incorporates also models whose behavior is characterized in terms of variational-hemivariational inequalities, see Section 6. More details and examples on hemivariational inequalities can be found in [13,14,18–20].

The main result of this paper is Theorem 11 that states the conditions for the existence of a weak solution to the stationary flow problem for a generalized Newtonian fluid. We note that our method of proof is different from those in the aforementioned papers; it exploits a surjectivity result for multivalued pseudomonotone operators. However, the existence result holds, under a smallness hypothesis, in the case $p \geq \frac{3d}{2+d}$ with $d = 2, 3$ and the problem to relax this hypothesis is left open. Furthermore, in Theorem 12, we deliver sufficient conditions for the uniqueness of solution in the case $d = 2$ and for $p \in [2, \infty)$. The uniqueness of solution for the case $d = 3$ and other values of p remains another interesting open problem.

The paper is organized as follows. In Section 2 we recall preliminary material. The physical setting for the flow problem together with its classical and weak formulations is provided in Section 3. In Section 4 and 5, the results on existence and uniqueness of weak solution are proved, respectively. Some examples of the constitutive laws and frictional boundary conditions are given in Section 6.

2. Preliminaries

In this section we recall the basic notation and preliminary results needed in the following sections. Details can be found in [21–23].

Let $(X, \|\cdot\|_X)$ be a Banach space and let X^* denote its topological dual. The notation $\langle\cdot,\cdot\rangle_{X^*\times X}$ stands for the duality pairing of X^* and X. The space X endowed with its norm and weak topology, is denoted by X and w-X, respectively. We put $\|S\|_X = \sup\{\, \|s\|_X \mid s \in S\,\}$ for any subset S of X. By $\mathcal{L}(E, F)$ we denote the class of linear and bounded operators from a Banach space E to a Banach space F.

We recall the following definitions for single-valued operators. An operator $A : X \to X^*$ is called bounded if it maps bounded sets of X into bounded sets of X^*. It is called monotone if $\langle Au - Av, u - v\rangle_{X^*\times X} \geq 0$ for all $u, v \in X$. An operator A is called hemicontinuous if the real valued function $t \to \langle A(u + tv), w\rangle_{X^*\times X}$ is continuous on $[0, 1]$ for any $u, v, w \in X$. An operator $A : X \to X^*$ is said to be pseudomonotone, if it is bounded and if $u_n \to u$ weakly in X and $\limsup\langle Au_n, u_n - u\rangle_{X^*\times X} \leq 0$ implies $\langle Au, u - v\rangle_{X^*\times X} \leq \liminf\langle Au_n, u_n - v\rangle_{X^*\times X}$ for all $v \in X$. Recall that if X is reflexive Banach space, then every bounded, hemicontinuous and monotone operator is pseudomonotone (cf. Proposition 27.6(a) in [23]).

The next definitions hold for multivalued operators. For a multivalued operator $A : X \to 2^{X^*}$, its domain, range and graph are defined by $D(A) = \{x \in X \mid Ax \neq \emptyset\}$, $R(A) = \cup\{Ax \mid x \in X\}$ and $Gr(A) = \{(x, x^*) \in X \times X^* \mid x^* \in Ax\}$, respectively. The inverse operator to A, denoted by $A^{-1} : X^* \to 2^X$, is defined by $A^{-1}x^* = \{x \in X \mid x^* \in Ax\}$ for $x^* \in X^*$. Multivalued operator $A : X \to 2^{X^*}$ is called monotone, if for all $(x, x^*), (y, y^*) \in Gr(A)$, we have $\langle x^* - y^*, x - y\rangle_{X^*\times X} \geq 0$. An operator A is called maximal monotone, if it is monotone and if $(x, x^*) \in X \times X^*$ is such that

$$\left\langle x^* - y^*, x - y\right\rangle_{X^*\times X} \geq 0 \quad \text{for all } (y, y^*) \in Gr(A),$$

then $(x, x^*) \in Gr(A)$. The latter is equivalent to saying that $Gr(A)$ is not properly contained in the graph of any other monotone multivalued operator.

The following result provides a useful criterion for maximal monotonicity (cf. Proposition 1.3.11 of [22]).

Theorem 1. *Let X be a Banach space and $A : X \to 2^{X^*}$ be a multivalued operator. If the following conditions hold*

(i) *A is monotone;*
(ii) *for all $v \in X$, the set Av is nonempty, weakly-$*$-closed and convex;*
(iii) *for all $u, v \in X$, $\lambda \to A(\lambda u + (1 - \lambda)v)$ has a closed graph in $[0, 1] \times X^*$, where X^* is endowed with the weak-$*$ topology;*

then A is a maximal monotone operator.

The following definitions of pseudomonotonicity and generalized pseudomonotonicity (cf. Definition 1.3.63 of [22]) will be useful in the next sections.

Definition 2. Let X be a reflexive Banach space. A multivalued operator $A : X \to 2^{X^*}$ is called pseudomonotone, if the following conditions hold