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Numerical scheme for a model of shallow water waves in (2 + 1)-dimensions

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ABSTRACT

A nonlinear difference scheme is considered for the two-dimensional Rosenau–Burgers equation. Some priori estimates, existence and uniqueness of the difference solution have been shown. A second-order convergence in the uniform norm and stability are proved. Also a convergent iterative algorithm is presented. All results are obtained without any restrictions on the meshsizes. At last numerical experiments are carried out to support the theoretical claims.

1. Introduction

A nonlinear wave phenomenon is an important area of scientific research. There are mathematical models which describe the dynamic of wave behaviors such as the KdV equation, the Rosenau equation, and many others. The KdV equation cannot explain the wave–wave and wave–wall interactions for the model of the dynamics of compact discrete systems. Therefore, Rosenau [1,2] presented the novel model, which is more suitable than the KdV equation, as follows:

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \tag{1.1}$$

The theoretical results on existence, uniqueness and regularity of the solution for (1.1) have been investigated by Park [3]. But it is difficult to find the analytical solution for (1.1). Since then, much work has been done on the numerical methods for (1.1), see [4–8] and the references therein.

The dynamics of dispersive shallow water waves is extensively studied by various known models. These are the Rosenau–KdV equation [9–12] and Rosenau-RLW equation [13–16], Rosenau–Kawahara equation [17,18], Rosenau–Kawahara-RLW equation [19,20], and many others.

On the other hand, for the further consideration of the dissipation in space for the dynamic system, such as the phenomenon of bore propagation and the water waves, the viscous term $-\alpha u_{xx}$ needs to be included:

$$u_t + u_{xxxxt} - \alpha u_{xx} + u_x + uu_x = 0. \tag{1.2}$$

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This equation is usually called the Rosenau–Burgers equation. The behavior of the solution to Eq. (1.2) has been well studied for the past years [21–23]. Several second-order accuracy nonlinear and linear difference schemes for the generalized Rosenau–Burgers equation have been restricted to one-dimensional (1D) in [24–29]. To our knowledge, there are no papers presenting numerical study of the Rosenau–Burgers equation in two dimensions by finite difference methods. Therefore, in this article, we consider the periodical boundary problem for the following version of 2D Rosenau–Burgers (RB) equation:

$$u_t + \Delta^2 u_t - \alpha \Delta u + \beta \nabla \cdot u + u \nabla \cdot u = 0, \quad (x, y) \in \mathbb{R}^2, \quad 0 < t \le T, \tag{1.3}$$

and initial condition

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathbb{R}^2, \tag{1.4}$$

subject to the (L_1, L_2)-periodic boundary conditions

$$u(x + L_1, y, t) = u(x, y, t), \quad u(x, y + L_2, t) = u(x, y, t), \quad 0 < t \le T, \tag{1.5}$$

where $\alpha > 0$, $\beta \in \mathbb{R}$, $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is the Laplacian operator, $\nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ and $u_0(x, y)$ is a given (L_1, L_2)-periodic function.

In this paper, we will prove that the difference scheme (1.3)–(1.5) is convergent with the convergence rate of order two in the uniform norm without any restriction on the meshsizes. A brief outline of the paper is as follows, Section 2 is devoted to the construction of difference scheme. Some priori estimates are presented in Section 3, one proves the existence of the difference solution in Section 4. The uniqueness is analyzed in Section 5, in Section 6, second-order error estimates in L^∞ and stability are derived. In Section 7, an iterative algorithm for the difference scheme with the proof of the convergence are given. In the next section, concluding remarks are discussed. At last section, some numerical examples are presented to prove the theoretical results.

2. Nonlinear finite difference scheme

To solve the periodic initial-value problem (1.3)–(1.5), one can restrict it on a bounded domain $\Omega = [0, L_1] \times [0, L_2]$. For a positive integer N, let time-step $\tau = \frac{T}{N}$, $t_n = n\tau$, $0 \le n \le N$, and $t_{n+\frac{1}{2}} = \frac{1}{2}(t_n + t_{n+1})$, $0 \le n \le N - 1$. Given temporal discrete function $\{V^n \,/\, 0 \le n \le N\}$, we denote

$$V^{n+\frac{1}{2}} = \frac{V^{n+1} + V^n}{2}, \quad \partial_t V^n = \frac{V^{n+1} - V^n}{\tau}.$$

We define a partition of $[0, L_1] \times [0, L_2]$ by the rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ with $x_i = ih$, $y_j = jh$, $i = 0, 1, 2, \ldots, M_1 := [\frac{L_1}{h}]$, $j = 0, 1, 2, \ldots, M_2 := [\frac{L_2}{h}]$. Denote

$$\Omega_h = \left\{(x_i, y_j) \,/\, 0 \le i \le M_1,\ 0 \le j \le M_2\right\}, \quad \Omega_\tau = \{t_n \,/\, 0 \le n \le N\}.$$

We define the space of periodic grid functions on Ω_h as:

$$\mathcal{V}_h = \left\{V = (V_{i,j})_{i,j\in\mathbb{Z}} \,/\, V_{i,j} \in \mathbb{R}, \quad V_{i+M_1,j} = V_{i,j}, \quad V_{i,j+M_2} = V_{i,j}, \quad i, j \in \mathbb{Z}\right\}.$$

For $V \in \mathcal{V}_h$, denote

$$\delta_{+x} V_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{h}, \qquad \delta_{+y} V_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{h},$$
$$\delta_{-x} V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{h}, \qquad \delta_{-y} V_{i,j} = \frac{V_{i,j} - V_{i,j-1}}{h},$$

$$\delta_{0x} V_{i,j} = \frac{V_{i+1,j} - V_{i-1,j}}{2h}, \qquad \delta_{0y} V_{i,j} = \frac{V_{i,j+1} - V_{i,j-1}}{2h},$$
$$\delta_x^2 V_{i,j} = \delta_{+x}\delta_{-x} V_{i,j}, \qquad \delta_y^2 V_{i,j} = \delta_{+y}\delta_{-y} V_{i,j},$$
$$\nabla_h V_{i,j} = (\delta_{0x} + \delta_{0y}) V_{i,j}, \qquad \Delta_h V_{i,j} = (\delta_x^2 + \delta_y^2) V_{i,j}, \qquad \Delta_h^2 V_{i,j} = \Delta_h(\Delta_h V_{i,j}).$$

For $U \in \mathcal{V}_h$ and $V \in \mathcal{V}_h$ define the inner product

$$(U, V)_h = h^2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} U_{i,j} \cdot V_{i,j},$$