Computers and Operations Research 99 (2018) 38–47

Contents lists available at ScienceDirect

Computers and Operations Research

journal homepage: www.elsevier.com/locate/cor

Robust optimization for non-linear impact of data variation

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A R T I C L E I N F O

Article history:
Received 4 April 2017
Revised 7 June 2018
Accepted 10 June 2018
Available online 15 June 2018

Keywords:
Linear programming
Non-linear programming
Robust optimization
Duality
Uncertainty

A B S T R A C T

We extend the Γ-robustness approach proposed by Bertsimas and Sim for Linear Programs to the case of non-linear impact of parameter variation. The seminal work considered protection from infeasibility over the worst-case variation of coefficients in a constraint, this variation being controlled by an uncertainty budget called Γ. When coefficients are non-linear functions of a parameter subject to uncertainty, we study a piecewise linear approximation of the function, and show that the subproblem of determining the worst-case variation can still be dualized despite the discrete structure of the piecewise linear function. We conduct numerical experiments on three different problems: Capital Budgeting, Generalized Assignment and Knapsack problems to analyze the trade-off between feasibility and objective value for the robust solution of the piecewise linear approximation compared to the nominal solution, and to a simpler binary approximation. Despite the piecewise approximation, the robust solution reveals to remain feasible over the 6800 runs performed in our experiments, with an average deterioration of the objective value of only a few percents.

1. Introduction

In this paper we adapt the Γ-Robustness paradigm developed by Bertsimas and Sim (2004) to the case where coefficients in constraints depend on data the variation of which impacts the coefficient in a non-linear way. This is the case if one considers demand as a function of price, the Net Present Value (NPV) of an investment project as a non-linear function of the discount rate, or the choice probability as the ratio of exponentials of utilities in a logit choice model (we will more formally detail these situations at the end of this introduction). We consider a generic linear program P of the form:

$$\max \sum_{j=1}^{n} c_j x_j \tag{1}$$

$$\text{s.t} \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, \ldots, m \tag{2}$$

$$x \geq 0 \tag{3}$$

In the classical Γ-robustness approach, for some constraints i, coefficients a_{ij} are subject to uncertainty and described by $a_{ij} = \bar{a}_{ij} + \hat{a}_{ij} z_{ij}$, where $\bar{a}_{ij}$ is the nominal value of the coefficient, $\hat{a}_{ij}$ is its maximum variation, and $z_{ij} \in [-1, 1]$. The framework developed in Bertsimas and Sim (2004) generates a robust counterpart for P that ensures feasibility of the robust solution whatever the variation of coefficients satisfying a given level of conservatism. This level of conservatism is controlled by a so-called uncertainty budget: the number of coefficients that can vary in constraint i is supposed not to exceed a budget Γ_i, i.e., $\Sigma_j |z_{ij}| \leq \Gamma_i$. This is consistent with the fact that, if for example coefficients are demand forecasts or orders, not all of them will vary at the same time. The Γ-robustness approach (see Bertsimas and Sim, 2004 for the construction of the robust counterpart of P) extends the more conservative approach of Soyster (1973) and has the advantage to be more tractable than the non-linear approach of Ben-Tal and Nemirovski (1999) based on ellipsoidal uncertainty, as the robust model remains an LP. However, it does not enable to capture in a straightforward way some specific kinds of problems or uncertainty structures. In the literature, to the best of our knowledge we can identify two families of extensions of Γ-robustness that deal with (i) time-dependent variations of coefficients, (ii) distribution of the uncertainty interval, and (iii) Right-Hand Side (RHS) uncertainty (a recent survey on robust optimization can be found in Gabrel et al., 2014):

(i) *Time-dependent coefficients and adaptive uncertainty budget* Γ. Bertsimas et al. (2013) and Lorca and Sun (2015) address the unit commitment problem in the energy sector. In their work, the authors tackle the issue of defining a proper uncertainty set

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https://doi.org/10.1016/j.cor.2018.06.011

for their problem. Bertsimas et al. use a set described by three parameters: a nominal value ($\bar{a}^t_{ij}$), a maximum deviation ($\hat{a}^t_{ij}$), and a budget of uncertainty (Γ^t) for each point in time. In this way, the uncertainty set adapts to the evolution of the problem through time. Lorca and Sun propose the use of *dynamic uncertainty sets*, where the uncertainty at each time period is a function of the uncertainty in previous stages. The authors use time series models to control the boundaries of the uncertainty sets for their problem. Poss (2014) presents an approach to robust combinatorial optimization where the budget of uncertainty is given by a function γ, rather than a constant Γ. Function γ adjusts the degree of conservatism according to the cardinality of the optimal solution, especially for problems where the optimal solution is sparse. The results can be extended to different budget functions better capturing the uncertainty arising from a specific context.

(ii) *Distribution of the uncertainty interval.* While the works previously cited mainly focus on the uncertainty budget Γ, Büsing and D'Andreagiovanni (2014) propose an extension on the distribution of the uncertainty set, which they call *multi-band uncertainty*. Since the deviation probability may vary along the uncertainty set, e.g. small deviations are more likely to occur than large ones, the proposed uniformity assumption could lead to an unrealistic uncertainty set. Büsing and D'Andreagiovanni (2014) propose to break the set into multiple narrower "bands", each with a customized Γ value, according to historical data. The uncertainty set is thus built so as to approximate the shape of the distribution of deviations. Similarly, in Düzgün (2012); Düzgün and Thiele (2010, 2015), the authors propose the use of multiple ranges for the uncertain parameters, limiting overly conservative solutions when the uncertainty set is too wide. Düzgün and Thiele (2015) bound as well the number of parameters that fall into each range (pessimistic view), but also bound the number of parameters that take the worst-case value within each range. The authors apply this setting to a project selection problem.

Let us mention that the concept of uncertainty budgets have also been used for problems with RHS uncertainty (e.g., Gabrel and Murat, 2010, Minoux, 2011, and Billionnet et al., 2014 for column-wise uncertainty and two-stage problems with recourse variables).

In this paper, we keep the Γ-robustness principle that consists of protecting against infeasibility over a controlled variation of parameters. However, we extend it to cases when the coefficients are non-linear functions of given data that are potentially subject to variations. For example in capacitated Facility Location problems, demands D_j of demand sources j often appear in the left-hand side of capacity constraints. If there is uncertainty on the market price of the product, then we should consider $D_j = f_j(p_j)$ where p_j is the price of the product at demand source j (demand sources may be located in different countries) and f_j is generally non-linear, concave decreasing.

Also, consider a Capital Budgeting problem with uncertainty on project values (see Meier et al., 2001 for a study on this topic). The capital budgeting problem consists in selecting a portfolio of investment projects maximizing the total net value of the portfolio while respecting resource constraints. Then the Net Present Value (NPV) of a project j is the sum of discounted cash-flows F_{jt} over the years $t = 0, \dots, T$, i.e., $NPV_j = \sum_{t=0}^{T} \frac{F_{jt}}{(1+a_j)^t}$, where a_j is the discount rate for project j. This discount rate r_j depends on several factors, some are exogenous like the interest rate of the country where project j is located, some are endogenous like the level of return the decision-maker wants to compensate her risk (the various projects being more or less risky). In any case, a_j is likely to be subject to imprecision or uncertainty and the NPV is clearly non-linear in a_j.

In Multinomial Logit (MNL) choice models (Mc Fadden, 1973), the probability p_{ij} of customer i choosing product j is expressed by $p_{ij} = e^{u_{ij}}/(\sum_k e^{u_{ik}})$ which is a non-linear function of utilities u_{ij} that are known to be hard to calibrate precisely (see Espinoza García and Alfandari, 2015 for an application to the location of new housing developments). Another example could be non-linear consumption functions depending on uncertain temperatures.

Note that several papers deal with non-linear robust optimization, but their goal is more to study the robust counterpart of a non-linear problem, i.e. the nominal problem comprises non-linear functions of the variables (see for example Ben-Tal et al., 2017; Ben-Tal and Nemirovski, 1998; Ben-Tal et al., 2002; Diehl et al., 2006; Houska and Diehl, 2013; Kawas and Thiele, 2011; Takeda et al., 2008; Zhang, 2007). This is not the case studied in this paper since our nominal problem is linear, only the impact of parameter variation is non-linear in our framework. A robust network design problem was studied in Pessoa and Poss (2015) which was a linear program with a quadratic dependency on uncertainty in a constraint. However, the solving approach was not based on a generic compact reformulation as we do, but on cutting planes to generate uncertainty scenarios on the fly. Also, we found a paper on a robust chance-constrained knapsack problem (Han et al., 2016) that has some similarities with our approach. Ellipsoidal uncertainty is considered in this paper, which provides a quadratic impact of data variation in the knapsack constraint. Although this paper considers a specific problem and ellipsoidal uncertainty, the approximation of their quadratic function can be seen as a special case of our generic approach.

The paper is organized as follows. Section 2 describes the robust problem and discusses its tractability, as the robust problem cannot be turned into a linear program. In Section 3, we approximate the non-linear variation with a binary approximation. A more refined piecewise linear function is proposed in Section 4. We show that the approximation of the robust counterpart is tractable using LP dualization for the worst-case subproblem. Section 5 is devoted to numerical experiments and analyzes the performance of our extended robustness approach and of the piecewise approximation. Section 6 concludes the paper.

2. Problem statement

We consider constraints of the form

$$\sum_j f_{ij}(a_{ij})x_j \leq b_i$$

where $f_{ij}(a_{ij})$ is non-linear. We use the notation $f_{ij}(a_{ij})$ instead of $a_{ij} = f_{ij}(p_{ij})$ where p_{ij} would be the varying parameter, in order to keep the usual notation $a_{ij} = \bar{a}_{ij} + z_{ij}\bar{a}_{ij}$ where a_{ij} is the varying parameter and z_{ij} is the deviation variable in the robust counterpart. We denote by P_f the original problem P where coefficients in the constraints are of the form $f_{ij}(a_{ij})$, where a_{ij} is the coefficient subject to uncertainty and functions f are not linear. For this variant P_f, the robust transformation proposed in Bertsimas and Sim (2004) would result in a non-linear problem. If we denote as in Bertsimas and Sim (2004) by $\Gamma_i \in \mathbb{N}$ the maximum total variation of coefficients a_{ij} in constraint i (uncertainty budget), the robust problem associated with P_f can be expressed as:

$$\max \quad \sum_{j=1}^{n} c_j x_j \tag{4}$$

$$s.t \quad \omega_i(x, \Gamma_i) \leq b_i \qquad \forall\ i = 1, \dots, m \tag{5}$$

$$x \geq 0 \tag{6}$$