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New reformulations of distributionally robust shortest path problem

Jianqiang Cheng [a], Janny Leung [b], Abdel Lisser [a,*]

[a] Laboratoire de Recherche en Informatique (LRI), Université Paris Sud - XI, Bât. 650, 91405 Orsay Cedex, France
[b] Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong

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ABSTRACT

This paper considers a stochastic version of the shortest path problem, namely the Distributionally Robust Stochastic Shortest Path Problem (DRSSPP) on directed graphs. In this model, each arc has a deterministic cost and a random delay. The mean vector and the second-moment matrix of the uncertain data are assumed to be known, but the exact information of the distribution is unknown. A penalty occurs when the given delay constraint is not satisfied. The objective is to minimize the sum of the path cost and the expected path delay penalty. As this problem is NP-hard, we propose new reformulations and approximations using a sequence of semidefinite programming problems which provide tight lower bounds. Finally, numerical tests are conducted to illustrate the tightness of the bounds and the value of the proposed distributionally robust approach.

1. Introduction

The Shortest Path (SP) problem is a well-known combinatorial optimization problem and has been extensively studied for the last few decades [3,8,10]. The objective of SP is to find a path with minimum distance or cost between two specified vertices of a given graph. In the deterministic SP problem, all the parameters are assumed to be known. However, due to different kinds of real life uncertainties, it may be difficult to specify the parameters precisely. Assuming deterministic values for parameters could lead to infeasibilities when the prescribed deterministic solution is implemented. One way to address this issue is robust optimization where the constraints involving random parameters are satisfied for all realizations of the random events (see, e.g., Soyster [29], Ben-Tal and Nemirovski [5]). Moreover, the random parameters are defined within a given uncertainty set. For a comprehensive overview on robust optimization, we refer the reader to the book by Ben-Tal et al. [4], the survey by Gabrel et al. [12] and references herein.

The robust shortest path problem has been widely studied. For instance, Yu and Yang [33] studied the robust shortest path problem in a layered network under two robustness criteria; they proved that the problem is NP-complete and devised a pseudo-polynomial algorithm. Gabrel et al. [13] proposed an integer linear program formulation for the studied robust shortest path and analyzed the theoretical complexity of the resulting problems.

An alternative to robust optimization is to model the problem as a stochastic optimization problem. The stochastic shortest path problem (SSPP) has also been widely studied in the past decades [15,18,20,22,24]. Provan [25] and Polychronopoulos and Tsitsiklis [26] studied expected shortest paths in networks where information on arc cost values is accumulated as the graph is being traversed, while Nikolova [23] maximized the probability that the path length does not exceed a given threshold value. Nie and Wu [22] studied the problem of finding a priori shortest paths to guarantee a given likelihood of arriving on-time in a stochastic network and also provided a pseudo-polynomial approximation based on extreme-dominance.

In transportation management systems, stochastic optimization has been applied widely as well. Sen et al. [27] formulated a network flow multiobjective model where one objective function consists in minimizing the expected travel-time between given origin and destination nodes whereas the second objective function minimizes the variance of travel-time. Miller-Hooks and Mahmassani [17] addressed the problem of determining least expected time paths in stochastic, time-varying networks where the arc weights (arc travel times) are random variables with probability distribution functions that vary with time. Xing and Zhou [32] investigated a fundamental problem of finding the most reliable path under different spatial correlation assumptions, and a Lagrangian substitution approach is used to get a lower bound. Fu and Rilett [11] studied a dynamic and stochastic shortest path problem to come-up with the expected shortest path in a traffic network where the link travel times are modeled as a continuous-time stochastic process,

* Corresponding author.
E-mail addresses: cheng@lri.fr (J. Cheng), janny@se.cuhk.edu.hk (J. Leung), lisser@lri.fr (A. Lisser).

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and proposed a heuristic algorithm based on the k-shortest path algorithm. In a recent paper, Mokarami and Hashemi [19] considered both robust and stochastic versions of the constrained shortest path problem, where an uncertain transit time was associated to each arc in addition to the arc cost. Moreover, they presented tractable approaches for solving the corresponding robust and stochastic constrained shortest path problems.

Most formulations and solution algorithms that address the SSPP require the knowledge of the underlying probability distributions of the random data. When the probability distribution is not known in advance, distributionally robust optimization can be used to handle the uncertainty [14] where only a part of the uncertainty information is required, such as the first two moments and the uncertainty support [7,9]. In addition, a wide range of distributionally robust optimization problems can be reformulated as SDP problems, and hence solved efficiently thanks to semidefinite programming (SDP) [9].

In this paper, we study the Distributionally Robust Stochastic Shortest Path Problem (DRSSPP) where only a part of the information on random data is assumed to be known. In this model, each arc has a deterministic cost and a random delay. Furthermore, we assume that only the first and the second moments of the delay are known. This problem has a simple recourse formulation, i.e., we deal with the delays of the path by introducing a penalty which is incurred when the delay constraint is not satisfied. The objective is to minimize the sum of the path cost and the expected path delay penalty. As the deterministic shortest path problem with delay is NP-hard [31], it follows that DRSSPP is also NP-hard by choosing all the arc variances equal to 0.

This paper is organized as follows. In Section 2, we give the mathematical formulation of DRSSPP. Two equivalent deterministic formulations are presented in Section 3. In Section 4, we present a copositive reformulation of DRSSPP when the support is nonnegative. In Section 5, two relaxed versions of DRSSPP are given to approximate the original problem. In Section 6, a numerical study is provided to evaluate the approximation and to illustrate the value of the proposed distributionally robust approach. The conclusions are given in the last section.

2. DRSSPP formulation

Let $\mathcal{G} = (V, A)$ be a digraph with $n = |V|$ nodes and $m = |A|$ arcs. Each arc $a \in A$ has an associated cost $c(a) > 0$ as well as a random delay represented by the random variable $\tilde{\delta}(a)$. We assume w.l.o.g that $c_1, \ldots, c_m$ denote the costs while $\tilde{\delta}_1, \ldots, \tilde{\delta}_m$ are the random delays. Let $c = \{c_1, \ldots, c_m\}$ and $\tilde{\delta} = \{\tilde{\delta}_1, \ldots, \tilde{\delta}_m\}$.

When the exact probability distribution of $\tilde{\delta}$ denoted by $\mathcal{F}$ is known, the *Stochastic Shortest Path Problem* (*SSPP*) consists in finding a directed path between two given vertices s and t such that the sum of the cost and the expected delay cost is minimal. The delay cost is based on a penalty per time unit $d > 0$ that has to be paid whenever the total delay exceeds a given threshold $D > 0$. In transportation applications, D may represent the preferred arrival time and d the unit cost of delay, so the last term of the objective represents the expected cost of delay.

Then, SSPP can be mathematically formulated as follows [6]:

$$\text{(SSPP)} \quad \min_{x \in \{0,1\}^m} \quad c^T x + d \cdot \mathbb{E}_{\mathcal{F}}\left[\tilde{\delta}^T x - D\right]^+ \tag{1a}$$

$$\text{s.t.} \quad Mx = b \tag{1b}$$

where $[\cdot]^+ = \max\{0, \cdot\}$, $\mathbb{E}[X]$ denotes the expectation of a random variable X, $M \in \mathbb{R}^{n \times m}$ is the *node-arc incidence matrix* and $b \in \mathbb{R}^n$, with all elements being 0 except the s-th and t-th elements, which are 1 and -1, respectively [1].

The objective function is composed of two terms, namely the total cost of the shortest path and the expectation cost related to the delay constraint. The second term can be interpreted as the expectation of individual penalization of excess delays of the arcs. This formulation is also known in stochastic programming as a simple recourse formulation.

2.1. Distributionally robust formulation

SSPP requires that the exact information of the distribution $\mathcal{F}$ is known. However, this is not often the case for many practical problems. Therefore, distributionally robust optimization can be used to handle the uncertainty. In this paper, we model the SSPP as distributionally robust SSPP as follows:

$$\text{(DRSSPP)} \quad \min_{x \in \{0,1\}^m} \quad c^T x + d \cdot \max_{\mathcal{F} \in \mathcal{D}} \mathbb{E}_{\mathcal{F}}\left[\tilde{\delta}^T x - D\right]^+ \tag{2a}$$

$$\text{s.t.} \quad Mx = b \tag{2b}$$

where $\mathcal{D}$ is the collection of probability distributions of interest.

In the following, DRSSPP is considered under the following key assumption:

Assumption (A1). The distributional uncertainty set accounts for information about the support $\mathcal{S}$, mean μ, and an upper bound Σ on the covariance matrix of the random vector $\tilde{\delta}$

$$\mathcal{D}(\mathcal{S}, \mu, \Sigma) = \left\{ \mathcal{F} \in \mathcal{M} \,\middle|\, \begin{array}{l} \mathbb{P}(\tilde{\delta} \in \mathcal{S}) = 1 \\ \mathbb{E}_{\mathcal{F}}[\tilde{\delta}] = \mu \\ \mathbb{E}_{\mathcal{F}}[(\tilde{\delta} - \mu)(\tilde{\delta} - \mu)^T] \preceq \Sigma \end{array} \right\}.$$

where $\mathcal{M}$ is the set of all probability distributions on the measurable space $(\mathbb{R}^m, \mathcal{B})$, with $\mathcal{B}$ being the Borel σ-algebra on $\mathbb{R}^m$.

3. Deterministic formulations

In this section, we present two equivalent deterministic formulations of DRSSPP. The first formulation is a direct derivation similar in approach to previous work [9]. The second one is a new formulation with a smaller matrix constraint size than the first one, which is much more effective computationally, as shown in Section 6.

Delage and Ye [9] have previously studied the distributionally robust approach; they gave an equivalent deterministic formulation which we apply hereafter to DRSSPP.

Theorem 1. *Under assumption (A1), together with $\mathcal{S} = \mathbb{R}^m$, problem (2) is equivalent to the following deterministic problem:*

$$\text{(DRSSPP1)}: \quad \min_{x \in \{0,1\}^m, t \in \mathbb{R}, \mathbf{q} \in \mathbb{R}^m, \mathbf{Q} \in \mathbb{R}^{m \times m}}$$

$$c^T x + d \cdot ((\Sigma + \mu\mu^T) \bullet \mathbf{Q} + \mu^T \mathbf{q} + t) \tag{3a}$$

$$\begin{bmatrix} t + D & \dfrac{(\mathbf{q} - x)^T}{2} \\ \dfrac{\mathbf{q} - x}{2} & \mathbf{Q} \end{bmatrix} \succeq 0 \tag{3b}$$