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Relaxation approach for equilibrium problems with equilibrium constraints

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ABSTRACT

We study a generalization of the relaxation scheme for mathematical programs with equilibrium constraints (MPECs) studied in Steffensen and Ulbrich (2010) [31] to equilibrium problems with equilibrium constraints (EPECs). This new class of optimization problems arise, for example, as reformulations of bilevel models used to describe competition in electricity markets. The convergence results of Steffensen and Ulbrich (2010) [31] are extended to parameterized MPECs and then further used to prove the convergence of the associated method for EPECs. Moreover, the proposed relaxation scheme is used to introduce and discuss a new relaxed sequential nonlinear complementarity method to solve EPECs. Both approaches are numerically tested and compared to existing diagonalization and complementarity approaches on a randomly generated test set.

1. Introduction

In 1950 Nash introduced in [21,22] the *noncooperative Nash game* which is used to model the situation where J players try to find their optimal strategy according to their strategy sets and their own pay-off functions. These, however, depend on the other players' chosen strategies. A strategy combination, where no player can improve his own strategy as long as all the other players stick to their strategy is called a *Nash equilibrium*. The so-called *Stackelberg game* [29] is similar to the Nash game, however, there exists one distinct player (the *leader*), that dominates all the other players (the *followers*) playing a Nash game among each other.

EPECs are a special variant of the general Nash game that can be regarded as an extension of the classical Stackelberg game, where more than just one distinct player exist (the *leaders*) and all N ($N>1$) leaders dominate a second set of players (the *followers*). Moreover, not only the followers, but also the leaders play a noncooperative Nash game among each other. In this sense EPECs might also be called *Bilevel games*, where each player of the upper-level Nash game has to solve a *Bilevel Program*. Hence, the problem is to find a Nash equilibrium of the leading players' strategies $\bar{x}=(\bar{x}_1,\dots,\bar{x}_N)$, where each of the leaders' strategies $\bar{x}_i$ ($i=1,\dots,N$) solves

$$\begin{aligned} \min_{x_i,y} \quad & f_i(x_i,\bar{x}_{-i},y) \\ \text{s.t.} \quad & x_i \in X_i(\bar{x}_{-i}), \\ & y \in \mathcal{S}(x_i,\bar{x}_{-i}), \end{aligned} \tag{1}$$

where $x_{-i} := (x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)$, i.e. x_{-i} denotes the vector of the other players' strategies, X_i denotes the strategy set of player i (that might also depend on the other players' strategies), y denotes the vector of the followers' strategies and $\mathcal{S}(x_i,\bar{x}_{-i})$ denotes the solution set of the followers' Nash game that is parameterized by the leaders' strategies. Often, the so-called equilibrium constraint $y\in\mathcal{S}(x_i,\bar{x}_{-i})$ can be replaced by a so-called variational inequality (for more informations on VIs see e.g. [9]). In this case (1) belongs to the class of *mathematical programs with equilibrium constraints* (MPECs) (cf. also the monographs on MPECs [20,17]).

If all feasible sets are determined by equalities and inequalities, i.e. the feasible set $X_i(\bar{x}_{-i})$ is given by

$$h_i(x_i,\bar{x}_{-i},y)=0 \quad \text{and} \quad g_i(x_i,\bar{x}_{-i},y)\geq 0$$

and the lower level problems of the $j=1,\dots,J$ followers are given by

$$\begin{aligned} \min_{y_j} \quad & F_j(x_i,\bar{x}_{-i},y) \\ \text{s.t.} \quad & C_h^j(x_i,\bar{x}_{-i},y)=0, \\ & C_g^j(x_i,\bar{x}_{-i},y)\geq 0, \end{aligned}$$

with appropriately chosen dimensions and $y=(y_1,\dots,y_J)$, then problem (1) is of the form of a standard Bilevel Program

$$\begin{aligned} \min_{x_i,y} \quad & f_i(x_i,\bar{x}_{-i},y) \\ \text{s.t.} \quad & g_i(x_i,\bar{x}_{-i},y)\geq 0, \\ & h_i(x_i,\bar{x}_{-i},y)=0, \end{aligned}$$

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$$y_j \text{ solves } \begin{cases} \min_{y_j} & F_j(x_i,\overline{x}_{-i},y) \\ \text{s.t.} & C_h^j(x_i,\overline{x}_{-i},y)=0, \quad \forall j=1,\ldots J \\ & C_g^j(x_i,\overline{x}_{-i},y)\geq 0, \end{cases} \tag{2}$$

(for more informations on Bilevel Programs see e.g. the monographs [1,3]). Hence the Bilevel game is the problem of finding a strategy vector $(\overline{x},\overline{y})$ that satisfies

$$(\overline{x}_i,\overline{y}) \text{ solves } \begin{cases} \min_{x_i,y} & f_i(x_i,\overline{x}_{-i},y) \\ \text{s.t.} & g_i(x_i,\overline{x}_{-i},y)\geq 0, \\ & h_i(x_i,\overline{x}_{-i},y)=0, \\ & y_j \text{ solves } (j=1,\ldots J) \begin{cases} \min_{y_j} & F_j(x_i,\overline{x}_{-i},y) \\ \text{s.t.} & C_h^j(x_i,\overline{x}_{-i},y)=0, \\ & C_g^j(x_i,\overline{x}_{-i},y)\geq 0, \end{cases} \end{cases}$$

$$\forall\, i=1,\ldots,N.$$

Such games are sometimes also called *multi-leader-(identical)-follower games* [10,19,30] and used for example to model the strategic behaviour of the participants in deregulated electricity markets [16,15,19].

Unfortunately, in many instances the existence of equilibria of these games is not guaranteed. Hence, one might consider a model and attempt to solve the associated equilibrium problem that might very well have no solution. Studies that examined such questions can e.g. be found in [10,18].

Under suitable regularity conditions, the *lower-level* problems, can be replaced by their first order optimality conditions. This approach then results in a so-called *equilibrium problem with equilibrium constraints* (EPECs) of the form (here to support ease of reading we already concatenated all KKT conditions of the lower level problems)

$$(\overline{x},\overline{y},\overline{u},\overline{v}) \text{ solves } \begin{cases} \min_{x_i,y,u_i,v_i} & f_i(x_i,\overline{x}_{-i},y,u_i,v_i) \\ \text{s.t.} & g_i(x_i,\overline{x}_{-i},y,u_i,v_i)\geq 0, \\ & h_i(x_i,\overline{x}_{-i},y,u_i,v_i)=0, \\ (y,u_i,v_i) \text{ solves} & \begin{cases} 0=\nabla_y F(x_i,\overline{x}_{-i},y) \\ \quad -\nabla_y C_g(x_i,\overline{x}_{-i},y)^T u_i \\ \quad +\nabla_y C_h(x_i,\overline{x}_{-i},y)^T v_i, \\ 0=C_h(x_i,\overline{x}_{-i},y), \\ 0\leq u_i \perp C_g(x_i,\overline{x}_{-i},y)\geq 0. \end{cases} \end{cases}$$

$$\forall i=1,\ldots,N \tag{3}$$

These problems are the main objective of this paper. In general, (3) is a relaxation of the original Bilevel game, since the first order optimality conditions are only necessary but not sufficient. However, if the lower-level problem is convex, both problems are equivalent, since the KKT-conditions are necessary and sufficient under suitable regularity assumptions.

To date only a few results on the theoretical level are available in the literature (see e.g. [23,2,13] and the references therein). Moreover, the research on suitable methods for EPECs has just begun. The majority of algorithms proposed in the literature to solve EPECs are based either on diagonalization techniques such as Gauss–Jacobi or Gauss–Seidel methods [15,19,30] or on the solution of a concatenated system of stationarity conditions. Since here only one single nonlinear complementarity problem (NCP) has to be solved these methods tend to be more efficient than the diagonalization methods [15,19]. However, both of these studies also report on disadvantages of the NCP approach, either in a lack of efficiency (in terms of CPU times and/or iteration numbers) for large scale problems [15] or in robustness [19], i.e. the percentage of (solvable) problems solved. Another very recent paper [25] studies the solution of an EPEC model for an oligopolistic electricity pool. Therein the EPEC is first reformulated as an NCP, however, due to its structure, the NCP can be further transformed into a mixed integer linear program (MILP) using a linearization without approximation. The resulting MILP is then solved by some standard Branch & Cut algorithm.

In this paper we propose a Gauss–Seidel method, where each single-leader–follower game denoted as $MPEC(x_{-i})$ is solved using the relaxation method for MPECs that is proposed in [31] and proved to be successful in solving MPECs numerically. The relaxation scheme is based on a reformulation of the complementarity constraints that is exact for the complementarity conditions corresponding to sufficiently non-degenerate complementarity components and relaxes only the remaining complementarity conditions. Moreover, a positive parameter determines to what extent the complementarity conditions are relaxed.

General assumptions and notations: We assume that all functions involved in the original problem (i.e. f_i, g_i, h_i, F, G,H) are twice continuously differentiable with respect to all their arguments (i.e. the variables x and y and the parameter vector a in Section 2). The operator ∇ denotes either the gradient or the transposed Jacobian of the corresponding scalar or vector valued function, respectively. We define the support of $\lambda\in\mathbb{R}^m$ as $\text{supp}(\lambda) := \{j\in\{1,\ldots,m\} : \lambda_j\neq 0\}$ and use $\mathbb{R}^n_+$ as notation for the nonnegative orthant of $\mathbb{R}^n$. Finally, we use the notation $(I_G\backslash I_H)(\overline{x})$ for $I_G(\overline{x})\backslash I_H(\overline{x})$ and $(I_G\cap I_H)(\overline{x})$ for $I_G(\overline{x})\cap I_H(\overline{x})$, respectively.

In the convergence analysis of our relaxation method we will use a variant of the following constraint qualification that was introduced for general NLPs in [24] and applied to MPECs in [12].

Definition 1.1. Let $\hat{x}$ be feasible for the standard nonlinear program

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & h(x)=0, \\ & g(x)\geq 0. \end{aligned}$$

1. Let $\mathcal{K}_g\subseteq I_g(\hat{x})$, $\mathcal{K}_h\subseteq I_h(\hat{x})$ (where $I_h(x)=\{i : h_i(x)=0\}$ and $I_g(x)=\{j : g_j(x)=0\}$). We call the family of gradient vectors
$\{\nabla g_j(\hat{x}) : j\in\mathcal{K}_g\}\cup\{\nabla h_j(\hat{x}) : j\in\mathcal{K}_h\}$
positive linearly dependent, if there exist vectors $\mu\in\mathbb{R}^{|\mathcal{K}_h|}$ and $\lambda\in\mathbb{R}^{|\mathcal{K}_g|}_+$ with $(\mu,\lambda)\neq(0,0)$ and
$$\sum_{i\in\mathcal{K}_h}\mu_i\nabla h_i(\hat{x})+\sum_{j\in\mathcal{K}_g}\lambda_j\nabla g_j(\hat{x})=0.$$

2. The CPLD (*constant positive linear dependence constraint qualification*) is said to hold in $\hat{x}$, if for every $\mathcal{K}_g\subseteq I_g(\hat{x})$, $\mathcal{K}_h\subseteq I_h(\hat{x})$, such that the family of gradient vectors
$\{\nabla g_j(\hat{x}) : j\in\mathcal{K}_g\}\cup\{\nabla h_j(\hat{x}) : j\in\mathcal{K}_h\}$
is positive linearly dependent, there exists a neighbourhood $\mathcal{U}(\hat{x})$, such that for every $y\in\mathcal{U}(\hat{x})$ the family
$\{\nabla g_j(y) : j\in\mathcal{K}_g\}\cup\{\nabla h_j(y) : j\in\mathcal{K}_h\}$
is linearly dependent.

1.1. Preliminaries on MPECs

As we briefly illustrated, under some suitable conditions the VI (or equilibrium constraint) of an MPEC can be replaced by a so-called complementarity problem, that consists of a system of (in)equalities and complementarity conditions. In this case the MPEC is also referred to as a *Mathematical program with*

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