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Innovative Applications of O.R.

Fluctuation identities with continuous monitoring and their application to the pricing of barrier options

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A B S T R A C T

We present a numerical scheme to calculate fluctuation identities for exponential Lévy processes in the continuous monitoring case. This includes the Spitzer identities for touching a single upper or lower barrier, and the more difficult case of the two-barriers exit problem. These identities are given in the Fourier-Laplace domain and require numerical inverse transforms. Thus we cover a gap in the literature that has mainly studied the discrete monitoring case; indeed, there are no existing numerical methods that deal with the continuous case. As a motivating application we price continuously monitored barrier options with the underlying asset modelled by an exponential Lévy process. We perform a detailed error analysis of the method and develop error bounds to show how the performance is limited by the truncation error of the sinc-based fast Hilbert transform used for the Wiener–Hopf factorisation. By comparing the results for our new technique with those for the discretely monitored case (which is in the Fourier-z domain) as the monitoring time step approaches zero, we show that the error convergence with continuous monitoring represents a limit for the discretely monitored scheme.

1. Introduction

Identities providing the Fourier-z transform of probability distribution functions of the extrema of a random path subject to monitoring at discrete intervals were first published by Spitzer (1956). They were extended to the continuous case by Baxter and Donsker (1957) and to double barriers by Kemperman (1963). The identities for the minimum and maximum of a path, for use with a single upper or lower barrier and for the two-barrier exit problem, are comprehensively described in the discrete monitoring case by Fusai, Germano, and Marazzina (2016), who proposed numerical methods to compute them for exponential Lévy processes. The discretely and continuously monitored identities are in the Fourier-z and Fourier-Laplace domains respectively. This means that, with the application of the inverse z or Laplace transform as appropriate, they can be used within Fourier transform option pricing methods, which we will use as an example in this paper. The relevance of the Spitzer identity in several fields within operational research is nowadays well recognised. We mention, for example, the application to queuing systems, see the classical contributions by Cohen (1975, 1982) and Prabhu (1974) and more recent work by Bayer and Boxma (1996), Markov chains (Rogers, 1994), insurance (Chi & Lin, 2011), inventory systems (Cohen & Pekelman, 1978; Grassmann & Jain, 1989), and applied probability (Grassman, 1990), as well as in mathematical finance.

Pricing derivatives, especially exotic options, is a challenging problem often covered also in the operations research literature; see e.g. Kou (2008). Fusai et al. (2016) provide extensive references for this, as well as for many non-financial applications of the Hilbert transform and the related topics of Wiener–Hopf factorisation and Spitzer identities in insurance, queuing theory, physics, engineering, applied mathematics, etc. Derivative pricing with Fourier transforms was first investigated by Heston (1993). Carr and Madan (1999) published the first method with both the characteristic function and the payoff in the Fourier domain. Fang and Oosterlee (2008, 2009) devised the COS method based on the

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Table 1
Characteristic exponent of some Lévy processes.

Process	Characteristic exponent $\psi(\xi)$	Rational
Normal	$i\xi\mu - \frac{1}{2}\sigma^2\xi^2$	✓
Kou	$i\xi\mu - \frac{1}{2}\sigma^2\xi^2 + \lambda\left(\frac{(1-\rho)\eta_2}{\eta_2 + i\xi} + \frac{\rho\eta_1}{\eta_1 - i\xi}\right)$	✓
Merton	$i\xi\mu - \frac{1}{2}\sigma^2\xi^2 + \lambda\left(e^{i\alpha\xi - \frac{1}{2}\delta^2\xi^2} - 1\right)$	✗
NIG	$\delta\left(\sqrt{\alpha^2 - (\beta + i\xi)^2} - \sqrt{\alpha^2 - \beta^2}\right)$	✗
VG	$-\frac{1}{\nu}\log\left(1 - i\xi\theta\nu + \frac{1}{2}\nu\sigma^2\xi^2\right)$	✗

Fourier-cosine expansion. Innocentis and Levendorskiĭ (2014) coupled piecewise polynomial interpolation with an efficient version of the Fourier transform technique. Kirkby (2017) exploited the frame-projected transition densities, which transform the problem into the Fourier domain and accelerate the convergence of intermediate expectations. The Hilbert transform (King, 2009) has also been successfully employed: by Feng and Linetsky (2008) to price barrier options using backward induction in Fourier space, and by Marazzina, Fusai, and Germano (2012) and Fusai et al. (2016) to compute via the Plemelj–Sokhotsky relations the factorisations required by the Wiener–Hopf method and the Spitzer identities. For a comparison of the two approaches in the discrete monitoring case, see Phelan, Marazzina, Fusai, and Germano (2018). Feng and Linetsky showed that computing the Hilbert transform with the sinc expansion, as studied by Stenger (1993, 2011), gives errors that reduce exponentially as the number of fast Fourier transform (FFT) grid points increases. However, the Feng and Linetsky method cannot be extended to continuously monitored options because its recursive structure makes it an inherently discrete scheme. In contrast (Green, Fusai, & Abrahams, 2010) showed that methods based on the Spitzer identities can be extended to continuous monitoring using the Laplace transform in the time domain rather than the z-transform. Unfortunately, they limited their analysis to the Gaussian case.

In this article we implement a method to numerically calculate the required Wiener–Hopf factors and thence the Spitzer identities in continuous time; we apply this to price continuously monitored options with general exponential Lévy processes. For continuous monitoring, the Wiener–Hopf factorisation can be done analytically if the characteristic exponent is rational (see Eq. (7) and Table 1), i.e. for the Gaussian and Kou double exponential processes, or in some special cases, e.g. when the jumps are only positive or negative. It is also possible to approximate an irrational exponent with a rational one that is easily factored (Kuznetsov, 2010). However, an analytical solution for the continuous monitoring case which is usable for any exponential Lévy process and does not require approximation has not been found yet. Therefore, the importance of our contribution is that it provides a formula to compute the Wiener–Hopf factors with a single barrier or two barriers in the continuous monitoring case. Moreover, we also propose a fast and accurate numerical method to make the computation of the Wiener–Hopf factors operational for any Lévy process, even when the exponent is not rational, like the variance gamma process. In the discrete case an analytical Wiener–Hopf factorisation can be done only for a Gaussian process (Fusai, Abrahams, & Sgarra, 2006), but from a numerical point of view the problem is easier and there are a number of papers dealing with exponential Lévy processes. However, it is well known that the convergence of numerical methods for discrete monitoring to the continuous monitoring limit is very slow; (see e.g. Broadie, Glasserman, & Kou, 1997). Therefore this work contributes to the literature by providing a procedure to determine the finite-time distribution of the extrema and of the hitting times in the presence of one or two barriers for a process with independent and identically distributed increments, such as a Lévy process, whereas previous numerical methods, like the one by Fusai et al. (2016), dealt only with discrete monitoring. Even if this article is mainly motivated by applications in option pricing, its relevance is very much beyond it. First-passage problems with models based on Markov processes are also ubiquitous in physical, biological, social, actuarial and other sciences. For example, our technique could be used to compute the ruin probability, i.e. the probability that a Lévy process takes value in a set A at a time $T > 0$ given that the process never falls below a barrier B in the interval [0, T], i.e. $P(X(T) \in A, \min_{t\in[0,T]} X(t) > B)$. This a classical problem in actuarial science and applied probability; see for example Klüppelberg, Kyprianou, and Maller, 2004. For applications in physics and biophysics, see e.g. the review by Bray, Majumdar, and Schehr (2013). Similar problems also arise in statistics, see for example the classical paper by Chernoff (1961), or in studying when a process reaches for the first time an adverse threshold state (a patient dies, or an industrial device breaks down).

This method follows the approach suggested by Green et al. (2010) and is based on the Fusai, Germano and Marazzina (FGM) method (Fusai et al., 2016) with spectral filtering (Phelan et al., 2018). While the latter method is for discrete monitoring and thus in the Fourier-z domain, here we operate in the Fourier-Laplace domain. Besides the discrete Fourier transform (DFT), or actually the fast Fourier transform (FFT), which is a standard technique, we also require a numerical inverse Laplace transform; for the latter we use an algorithm proposed by Abate and Whitt (1992a, 1995), which is based on a Fourier series and is derived in a similar way to their well established numerical inverse z-transform (Abate & Whitt, 1992b). Spectral filters are a powerful technique to improve Fourier-based option pricing, introduced to this field by Ruijter, Versteegh, and Oosterlee (2015). Cui, Kirkby, and Nguyen (2017) and Phelan et al. (2018) showed that multiplying the Fourier input by a spectral filter speeds up the price convergence when the characteristic functions decays slowly. At the end, the error convergence of our procedure is slightly worse than first-order polynomial; we explain this in detail with reference to the truncation error of the sinc-based discrete Hilbert transform. Our results show that the error convergence is consistent with the error bound and the performance of the discretely monitored technique as the monitoring interval goes to zero.

The structure of this paper is as follows. In Section 2 we briefly run through Fourier, Hilbert, Laplace and z transforms and explain how they are used for the calculation of the Spitzer identities. We then present a numerical pricing scheme for continuously monitored options and explain its relationship with the FGM pricing scheme for discrete monitoring. Section 3 provides a discussion of the error convergence of the pricing technique with special reference to the truncation error of the sinc-based Hilbert transform. Section 4 shows the results that were achieved, comparing them with the results for the FGM method for discretely monitored options.

2. Fourier transform methods for option pricing

In this paper we make extensive use of the Fourier transform (see e.g. Kreyszig, 2011; Polyanin & Manzhirov, 1998), an integral transform with many applications. Historically, it has been widely employed in spectroscopy and communications, therefore much of the literature refers to the function in the Fourier domain as its spectrum. According to the usual convention in the financial literature, the forward and inverse Fourier transforms are defined as

$$\widehat{f}(\xi) = \mathcal{F}_{x\to\xi}[f(x)] = \int_{-\infty}^{+\infty} e^{i\xi x} f(x)dx, \qquad (1)$$