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Innovative Applications of O.R.

Optimal hazard models based on partial information

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ABSTRACT

The proportional hazards (PH), mixture hazards (MH), proportional reversed hazards (PRH), and mixture reversed hazards (MRH) models are widely used in various applications in many fields, but their optimality properties remain unknown. We represent these important reliability models as general escort models and derive them as solutions to several information theoretic formulations. Thus far, the escort models are defined in physics by the normalized powers of one or product of two probability mass or density functions and have been derived in statistics and physics as the solutions to different information formulations. The general escort models introduced in this paper include the escorts of densities, as well as the escorts of survival functions and cumulative distribution functions which represent the hazards models. Moreover, we show that the MH and MRH models are also optimal according to formulations in terms of the mean variation distance. Additional results explore reliability properties of the escort of two densities. A notable property is that the escort of two densities with non-constant hazard rates can be a constant hazard rate model. Another result characterizes the PH model in terms of the survival function of the escort of two densities. Comparisons of the MH, escort of two densities, and the mixture of two distributions are illustrated.

1. Introduction

Hazard models are widely used in various problems in many fields, including several areas of operations. The proportional hazards (PH) model, popularized by Cox (1972), is the most widely used reliability model in applications including, customer attrition analysis (Van den Poel & Lariviére, 2004), preventive maintenance scheduling (Kobbacy, Fawzi, Percy, & Ascher, 1997), optimal replacement (Makis & Jardine, 1992), to name a few. Mixture hazards (MH) model is used in reliability analysis, as the Bayesian predictive hazard rate (Lynn & Singpurwalla, 1997), and in the non-Bayesian context (Finkelstein, 2009). The proportional reversed hazards (PRH) and mixture reversed hazards (MRH) models are used in reliability analysis, see for example, Di Crescenzo (2000), Badía, Berrade, and Campos (2002), Finkelstein (2002, 2003), Sankaran and Gleenja (2011), Gupta and Gupta (2007), Li and Da (2010), Li, Da, and Zhao (2010), and Kayid, Al-nahawati, and Ahmad (2011). In some problems these models appear in a natural way. For instance the PH model describes the proportional impact of environment on the hazard rate. But in general these models are utilized for the ease of analysis and inference purposes. We will show that these models arise as optimal solutions according to information criteria based on partial information about the probability distribution of the underlying random variable.

Frequently, information is available about some characteristics of a random variable which can be formulated in terms of partial information that do not determine a unique probability distribution. When the partial information can be formulated in terms of a set of some constraints, a unique model can be found for the random variable via the information theory. In this approach probability models are derived by assigning probability closest to a given reference distribution subject to a set of constraints. The objective function is an information divergence. The most well-known information divergence is the Kullback–Leibler (KL) information measure. This measure is used in many fields for various purposes, including operations and decisions problems (see, for example, Alwan, Ebrahimi, & Soofi, 1998; Plischke, Borgonovo, & Smith, 2013; Saghafian & Tomlin, 2016). Minimization of the KL measure subject to constraints that encapsulate partial information is known as the minimum discrimination information (MDI), minimum cross-entropy, and minimum relative entropy. When the reference distribution is uniform, the information-theoretic

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approach is the maximum entropy (ME) principle which is an extension of Laplace's principle of insufficient reason for assigning probabilities. The ME distribution is most "non-committal" to information other than that explicitly taken into account via the constraints (Jaynes, 1957; 1982). Recent applications of the maximum entropy procedure to operations research problems include, for example, Anderson, Jörnsten, Sandal, Neonas, and Ubœ (2013), Asadi, Ebrahimi, Soofi, and Zarezadeh (2014), and Fleischhacker and Fok (2015).

Many well-known parametric families of univariate and multivariate probability distributions have been characterized as ME models subject to moment constraints; see, for example, Alwan et al. (1998). The ME characterizations have led to diagnostics for model fit and tests of distributional hypotheses (see Mazzuchi, Soofi, & Soyer, 2008 and references therein). Minimum dynamic discrimination information and maximum dynamic entropy models subject to partial information about hazard rate and mean residual functions also have been derived by Asadi, Ebrahimi, Hamedani, and Soofi (2004, 2005). The dynamic information measures consider the distributions of residual life or past life at the current age t of the item. In general, dynamic information measures are functions of t. It has been shown that the KL discrimination information between two residual distributions is free from t if and only if the lifetime distributions are PH (Ebrahimi & Kirmani, 1996). This information characterization of the PH models also holds in terms of the Rényi information measure (Asadi, Ebrahimi, & Soofi, 2005). However, the information formulations that provide the important non-parametric classes of hazard models such as the PH, MH, PRH, and MRH models as solutions are not known yet. We advance such information formulations for these important models using their common defining property, namely, these models give probabilities of outcomes of a random variable X in terms powers of one or more baseline probability distributions.

In physics literature, probability models constructed by the normalized powers of univariate mass and density functions are called *escort* distributions. The escort distribution of order q is defined by the normalized power of a probability density function (PDF) relative to a measure ν as

$$f(x) \stackrel{def}{=} \frac{f_1^q(x)}{\int_{\mathcal{X}} f_1^q(x) d\nu(x)}, \quad q > 0, \tag{1}$$

provided that the integral is finite. The generalized escort probability distribution of order q is defined by the normalized product of powers of two densities relative to a measure ν as follows:

$$f(x) \stackrel{def}{=} \frac{f_1^q(x) f_2^{1-q}(x)}{\int_{\mathcal{X}} f_1^q(x) f_2^{1-q}(x) d\nu(x)} = C_q f_1^q(x) f_2^{1-q}(x), \quad q \ge 0, \tag{2}$$

where C_q is the normalizing factor, provided that f_1 is absolutely continuous relative to f_2 and the integral is finite. Bercher (2012) introduced (2) as an escort model and interpreted it as the "geometric mean" of $f_i, i = 1, 2$ which as a function of q defines a curve, called the escort path. The escort distributions "have been used as an operational tool in the context of multifractals ... with interesting connections with the standard thermodynamics ... also prove to be useful in source coding where they enable to derive optimum codewords ..." (Bercher, 2012). Density in the form of (2) has appeared in statistics literature for decades with different names and nameless. Kullback (1959) derived the PDF (2) as the immediate illustration of the MDI theorem without giving it a name. The power prior distribution proposed by Ibrahim and Chen (2000) is in the form of (2) with $0 < q < 1$. Bercher (2012) derived (1) and (2) for some physics problems using information theoretic formulations which are different from those used by Kullback (1959) and Ibrahim, Chen, and Sinha (2003).

The objectives of this paper are as follows. First we address the following question: Are formulations of Kullback (1959), Ibrahim et al. (2003), and Bercher (2012) adaptable to provide the PH, PRH, MH, and MRH models as the information optimal solutions? The question is intricate and we explore some possibilities. We introduce a more general notion of escorts of probability models which includes the escort models (1) and (2) for the densities/mass functions, as well as the corresponding escort models for the survival functions (SFs) and the cumulative distribution functions (CDFs). The PH, MH, PRH, and MRH models are examples of the escort models of SFs and CDFs. We show that the PH, PRH, MH, and MRH models are optimal solutions to several information theoretic and variation formulations. These formulations are three types:

1. The classical MDI and ME formulations where the objective functions are in terms of densities, but the constraints are in terms of SFs or CDFs. These formulations show that the formulation of Kullback (1959) is adaptable to provide the PH and PRH models as the information optimal solutions. The results can lead to developing diagnostics for model fit and tests of distributional hypotheses based on partial information about the SF and CDF.
2. Formulations where the information functions may represent the usual KL function between two densities as well as the KL type measure between two SFs or two CDFs. These formulations show that Ibrahim et al. (2003) and Bercher (2012) are adaptable to provide the PH, PRH, MH, and MRH models as the information optimal solutions. These formulations establish that derivations of models based on partial information are not limited to the usual use of information measures for probability densities. These novel information formulations in terms of the SFs (or CDFs) possess estimation advantage over the usual information theoretic formulations in terms of densities, due to the fact that estimation of SFs (or CDFs) are simpler statistical tasks as compared with the density estimation.
3. Formulations in terms of variation function between cumulative hazard functions (− log of SFs). These formulations are also novel as they introduce the expected variation function as a new objective function for deriving probability models based on partial information.

Our second objective is to explore the reliability properties of the escort distributions constructed by two PDFs. The PDF (2) provides a bona fide probability model for combining two distributions for applications in various contexts. However, beside the information optimality, distributional properties of escorts of probability densities remain unexplored. We give a few properties of (2), including results for its hazard rate properties. An important property is that (2) can produce constant hazard rate models without the two constituent PDFs being exponential. We also characterize the PH model in terms of SF of the escort of two densities.

This paper is organized as follows. Section 2 gives an overview of the existing information formulations for derivations of (2) and defines the general escort models which include (1), (2), the PH, PRH, MH, and MRH models. Section 3 generalizes the existing formulations in terms of the general escort models which characterize the PH, PRH, MH, and MRH models as the optimal solutions. Section 4 presents formulations in terms of the mean variation distance which provide the MH and MRH models as the optimal solution. Section 5 gives the reliability properties of the escort PDF (2). Section 6 illustrates distinctions between the escort of two PDFs, the escort mixture of two SFs (mixture hazards), and the mixture of distributions (PDFs or SFs). Section 7 summarizes the paper and gives some concluding remarks.

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