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Decision Support

Multiplicative aggregation of division efficiencies in network data envelopment analysis

Chiang Kao

Department of Industrial and Information Management, National Cheng Kung University, Tainan, Taiwan

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ABSTRACT

Network systems have two basic structures, series and parallel, and a network system can be transformed into a series system of subsystems, where each has a parallel structure composed of a number of divisions. The efficiency of the system can thus be expressed as the product of those of the subsystems, and the efficiency of each subsystem is a weighted average of those of its component divisions, under a relational data envelopment analysis (DEA) model. A previous study showed that the efficiency of a network system can be expressed as an additive aggregation of those of the divisions adjusted by a factor, and the former is bounded from above by the latter. This paper shows that the efficiency of the system can be expressed as a multiplicative aggregation of those of the divisions multiplied by a factor of greater than one. The system efficiency is thus bounded from below by the multiplicative aggregation of the division efficiencies. An example is used to show how to calculate the efficiency bounds, and how to identify the divisions that have stronger effects on the performance of the system.

1. Introduction

Data envelopment analysis (DEA) is a technique for measuring the relative efficiency of a set of decision making units (DMUs) that apply multiple inputs to produce multiple outputs (Banker, Charnes, & Cooper, 1984; Charnes, Cooper, & Rhodes, 1978), and network DEA refers to applying the DEA technique to DMUs of network production systems composed of a number of divisions operating interdependently with each other (Färe & Grosskopf, 2000; Kao, 2017a). There are two major concepts for measuring efficiencies in network DEA, efficiency decomposition and aggregation (Kao, 2016). The concept of efficiency decomposition measures efficiency from the viewpoint of outside peers, in that the efficiency of the system is defined by the inputs and outputs of the system that are observable from outside, and a relationship between the system and division efficiencies is sought. The system efficiency is used for comparing the performance among DMUs. The concept of efficiency aggregation, in contrast, measures efficiency from the viewpoint of inside managers, in that the efficiency of the system is defined as an aggregation of those of the component divisions, and the efficiency of a division is defined by the inputs and outputs of this division. The system efficiency shows the aggregate performance of the divisions. The difference between these two is that the intermediate products produced and consumed within the system, which are not visible from outside, are also considered in measuring the system efficiency in efficiency aggregation. These two concepts lead to different efficiency measures for a system, and which one to use depends on the purpose of the measurement.

Kao (2016) investigated the relationship between the efficiencies measured from these two concepts. The way of aggregating the division efficiencies discussed in this paper is a weighted average, where the weight associated with each division efficiency is the proportion of the aggregate input consumed by this division in that consumed by all divisions. It was shown that the efficiency measured from the viewpoint of outside peers is less than or equal to the additive-aggregated efficiency, measured from the viewpoint of inside managers. However, it should be noted that additive aggregation is not the only method of aggregation, and multiplication is another common approach that has been applied to series systems (see, for example, Chen, Liang, & Yang, 2006; Zha & Liang, 2010; and Li, Chen, Liang, & Xie, 2012). The current paper explores the relationship between the system efficiency measured from the viewpoint of outside peers and that measured from the viewpoint of inside managers using a multiplicative aggregation of the division efficiencies. In contrast to the upper bound set by the additive-aggregated efficiency, as was revealed in Kao (2016), this paper shows that the multiplicative-aggregated efficiency serves as a lower bound for the system efficiency.

In the following sections, the method of measuring the system efficiency and aggregate efficiencies using a relational model is first reviewed. How the system efficiency is expressed as a function of

E-mail address: ckao@mail.ncku.edu.tw

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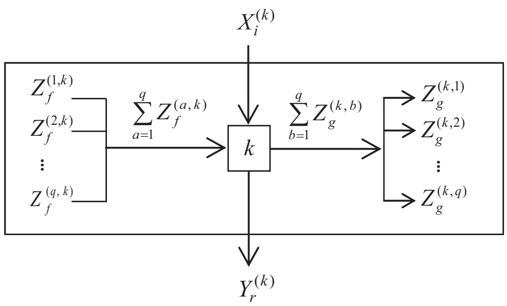

Fig. 1. General network system.

the division efficiencies is then discussed. Based on this, a lower bound for the system efficiency is derived, and an example is used for illustration. Finally, some conclusions are drawn based on the discussion in the preceding sections.

2. Relational model

Consider a general network system composed of q divisions, where each division k consumes the exogenous inputs $X_i^{(k)}$ supplied from outside and the endogenous inputs $Z_f^{(a,k)}$ produced by Division a to produce the exogenous outputs $Y_r^{(k)}$ to send out of the system and the endogenous outputs $Z_g^{(k,b)}$ for Division b to use. Fig. 1 depicts the structure of the general network system.

The inputs and outputs that are observable from outside are $X_i^{(k)}$ and $Y_r^{(k)}$. Each division of a DMU may not consume all the inputs X_i, $i=1,\ldots, m$, nor produce all the outputs Y_r, $r=1,\ldots, s$. For simplicity of expression, we set $X_{ij}^{(k)}=0$ and $Y_{rj}^{(k)}=0$ for Division k of DMU j when input X_i is not used and output Y_r is not produced. Likewise, we set $Z_{gj}^{(a,b)}=0$ if Division a of DMU j does not produce the intermediate products Z_g, $g=1,\ldots, h$, for Division b to use. The efficiency of the system of a DMU, as viewed by outside peers, is the ratio of the aggregation of the outputs observable from outside to that of the inputs also observable from outside. The constraints are that the aggregate output of each division does not exceed its aggregate input. The inputs and outputs of a division include endogenous ones, in addition to the exogenous ones. The relational model proposed by Kao (2009) for measuring the efficiency of a system under constant returns to scale in this case, as formulated in Kao (2017a), is thus:

$$
\begin{aligned}
E_0 = \max.& \sum_{k=1}^{q}\sum_{r=1}^{s} u_r Y_{r0}^{(k)} \\
\text{s.t.}& \sum_{k=1}^{q}\sum_{i=1}^{m} v_i X_{i0}^{(k)} = 1 \\
& \left[\sum_{r=1}^{s} u_r Y_{rj}^{(k)} + \sum_{g=1}^{h} w_g\left(\sum_{b=1}^{q} Z_{gj}^{(k,b)}\right)\right] \\
& -\left[\sum_{i=1}^{m} v_i X_{ij}^{(k)} + \sum_{f=1}^{h} w_f\left(\sum_{a=1}^{q} Z_{fj}^{(a,k)}\right)\right] \le 0, \quad k=1,\ldots,q, \\
& j=1,\ldots,n \\
& u_r,\ v_i,\ w_g \ge \varepsilon, \quad \forall r,i,g
\end{aligned}
\tag{1}
$$

where ε is a small non-Archimedean number imposed on the multipliers to avoid ignoring any factor in measuring efficiency (Charnes & Cooper, 1984). The feature of the relational model is that the same factor has the same multiplier associated with it, no matter which division it corresponds to.

When a set of optimal solutions is obtained, the system efficiency E_0 is the value of the objective function and the division efficiencies $E_0^{(k)}$ are calculated from the constraints, as:

$$
\begin{aligned}
E_0 &= \sum_{k=1}^{q}\sum_{r=1}^{s} u_r Y_{r0}^{(k)} \Big/ \sum_{k=1}^{q}\sum_{i=1}^{m} v_i X_{i0}^{(k)} = \sum_{k=1}^{q}\sum_{r=1}^{s} u_r Y_{r0}^{(k)} \\
E_0^{(k)} &= \left[\sum_{r=1}^{s} u_r Y_{r0}^{(k)} + \sum_{g=1}^{h} w_g\left(\sum_{b=1}^{q} Z_{g0}^{(k,b)}\right)\right] \\
&\quad \Big/ \left[\sum_{i=1}^{m} v_i X_{i0}^{(k)} + \sum_{f=1}^{h} w_f\left(\sum_{a=1}^{q} Z_{f0}^{(a,k)}\right)\right], \quad k=1,\ldots,q
\end{aligned}
\tag{2}
$$

Different from the outside peers who only see the exogenous inputs X_i supplied from outside and the exogenous outputs Y_r sent out of the system, the internal managers also see the intermediate products Z_g produced and consumed within the system. The inside managers will thus define the aggregation of the division efficiencies, $F(E_0^{(1)}, E_0^{(2)}, \ldots, E_0^{(q)})$, to be the system sufficiency. The corresponding model is:

$$
\begin{aligned}
\tilde{E}_0 = \max.&\ F\left(E_0^{(1)}, E_0^{(2)}, \ldots E_0^{(q)}\right) \\
\text{s.t.}& \left[\sum_{r=1}^{s} u_r Y_{rj}^{(k)} + \sum_{g=1}^{h} w_g\left(\sum_{b=1}^{q} Z_{gj}^{(k,b)}\right)\right] \\
& -\left[\sum_{i=1}^{m} v_i X_{ij}^{(k)} + \sum_{f=1}^{h} w_f\left(\sum_{a=1}^{q} Z_{fj}^{(a,k)}\right)\right] \le 0, \quad k=1,\ldots,q, \\
& j=1,\ldots,n \\
& u_r,\ v_i,\ w_g \ge \varepsilon, \quad \forall r,i,g
\end{aligned}
$$

This model has the same constraints as that of Model (1). At optimality, the system efficiency is the value of the objective function, and the division efficiencies $E_0^{(k)}$ are calculated from the constraints, which have the same expression as those defined in Eq. (2). To distinguish the system efficiencies measured from the viewpoints of outside peers and inside managers, hereafter that of the former will be simply called system efficiency and that of the latter will be called aggregated efficiency.

The most common way for aggregating the division efficiencies to form the system efficiency is the weighted average method (Cook, Zhu, Bi, & Yang, 2010). The idea is to use the proportion of the aggregate input consumed by a division in that consumed by all divisions as the weight for this division. In symbols, the weight is:

$$
\begin{aligned}
p^{(k)} &= \left[\sum_{i=1}^{m} v_i X_{i0}^{(k)} + \sum_{f=1}^{h} w_f\left(\sum_{a=1}^{q} Z_{f0}^{(a,k)}\right)\right] \\
&\quad \Big/ \sum_{k=1}^{q}\left[\sum_{i=1}^{m} v_i X_{i0}^{(k)} + \sum_{f=1}^{h} w_f\left(\sum_{a=1}^{q} Z_{f0}^{(a,k)}\right)\right]
\end{aligned}
$$

with $\sum_{k=1}^{q} p^{(k)}=1$ and $p^{(k)}\ge 0$. The additive-aggregated efficiency is thus:

$$
\begin{aligned}
F\left(E_0^{(1)}, E_0^{(2)}, \ldots, E_0^{(q)}\right) &= \sum_{k=1}^{q} p^{(k)} E_0^{(k)} \\
&= \sum_{k=1}^{q}\left(\frac{\sum_{i=1}^{m} v_i X_{i0}^{(k)} + \sum_{f=1}^{h} w_f(\sum_{a=1}^{q} Z_{f0}^{(a,k)})}{\sum_{k=1}^{q}[\sum_{i=1}^{m} v_i X_{i0}^{(k)} + \sum_{f=1}^{h} w_f(\sum_{a=1}^{q} Z_{f0}^{(a,k)})]}\right. \\
&\quad \left.\times \frac{\sum_{r=1}^{s} u_r Y_{r0}^{(k)} + \sum_{g=1}^{h} w_g(\sum_{b=1}^{q} Z_{g0}^{(k,b)})}{\sum_{i=1}^{m} v_i X_{i0}^{(k)} + \sum_{f=1}^{h} w_f(\sum_{a=1}^{q} Z_{f0}^{(a,k)})}\right) \\
&= \frac{\sum_{k=1}^{q}\left[\sum_{r=1}^{s} u_r Y_{r0}^{(k)} + \sum_{g=1}^{h} w_g\left(\sum_{b=1}^{q} Z_{g0}^{(k,b)}\right)\right]}{\sum_{k=1}^{q}\left[\sum_{i=1}^{m} v_i X_{i0}^{(k)} + \sum_{f=1}^{h} w_f\left(\sum_{a=1}^{q} Z_{f0}^{(a,k)}\right)\right]}
\end{aligned}
\tag{3}
$$