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Continuous Optimization

Compromise solutions for robust combinatorial optimization with variable-sized uncertainty

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ABSTRACT

In classic robust optimization, it is assumed that a set of possible parameter realizations, the uncertainty set, is modeled in a previous step and part of the input. As recent work has shown, finding the most suitable uncertainty set is in itself already a difficult task. We consider robust problems where the uncertainty set is not completely defined. Only the shape is known, but not its size. Such a setting is known as variable-sized uncertainty.

In this paper, we present an approach how to find a single robust solution, that performs well on average over all possible uncertainty set sizes. We demonstrate that this approach can be solved efficiently for min–max robust optimization, but is more involved in the case of min–max regret, where positive and negative complexity results for the selection problem, the minimum spanning tree problem, and the shortest path problem are provided. We introduce an iterative solution procedure, and evaluate its performance in an experimental comparison.

1. Introduction

Classic optimization settings assume that the problem data are known exactly. Robust optimization, like stochastic optimization, instead assumes some degree of uncertainty in the problem formulation. Based on the seminal papers (Ben-Tal & Nemirovski, 1998, 2000, 2002), most approaches in robust optimization formalize this uncertainty by assuming that all uncertain parameters ξ are described by a set of possible outcomes $\mathcal{U}$, the uncertainty set.

For general overviews on robust optimization, we refer to Ben-Tal, Ghaoui, and Nemirovski (2009), Bertsimas, Brown, and Caramanis (2011), and Gabrel, Murat, and Thiele (2014). Other surveys focus on robust combinatorial optimization (Aissi, Bazgan, & Vanderpooten, 2009; Kasperski & Zieliński, 2016), algorithmic developments (Goerigk & Schöbel, 2016) or present tutorials to the field (Chassein & Goerigk, 2016; Gorissen, Yanıkoğlu, & Hertog, 2015).

While the discussion of properties of the robust problem for different types of uncertainty sets $\mathcal{U}$ has always played a major role in the research community, only recently the data-driven design of useful sets $\mathcal{U}$ has become a focus of research. In Bertsimas, Gupta, and Kallus (2013), the authors discuss the design of $\mathcal{U}$ taking problem tractability and probabilistic guarantees of feasibility into account. Bertsimas and Brown (2009) discuss the relationship between risk measures and uncertainty sets, and Yanıkoğlu and Hertog (2012) constructs uncertainty sets by data-driven approximations of ambiguous chance constraints.

In distributionally robust optimization, one assumes that a probability distribution on the data is roughly known; however, this distribution itself is subject to an uncertainty set $\mathcal{U}$ of possible outcomes (see Ben-Tal, Hertog, de Waegenaer, Melenberg, & Rennen, 2013; Goh & Sim, 2010; Wiesemann, Kuhn, & Sim, 2014).

Another related approach is the globalized robust counterpart, see Ben-Tal et al. (2009). The idea of this approach is that a relaxed feasibility should be maintained, even if a scenario occurs that is not specified in the uncertainty set. The larger the distance of ξ to $\mathcal{U}$, the further relaxed becomes the feasibility requirement of the robust solution.

In this paper, we present an alternative to constructing a specific uncertainty set $\mathcal{U}$. Instead, we only assume knowledge of a nominal (undisturbed) scenario, and consider a set of possible uncertainty sets of varying size based on this scenario. That is, a decision maker does not need to determine the size of uncertainty, but only its shape. Our goal is to construct a solution for which the worst-case objective with respect to any possible uncertainty

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set performs well on average over all uncertainty sets parameterized this way.

The general idea of variable-sized uncertainty that this paper is based upon was recently introduced in Chassein and Goerigk (2018). There, the aim is to construct a set of robust candidate solutions that requires the decision maker to chose one that suits him best. This is inspired by parametric optimization, where one traces the change in the optimal solution while problem parameters vary. In our setting, we consider all uncertainty sizes simultaneously, and generate a single solution as a compromise approach to the unknown uncertainty. It can hence be regarded as a goal programming approach for variable-sized uncertainty. We call this setting the *compromise approach to variable-sized uncertainty*.

We focus on combinatorial optimization problems with uncertainty in the objective function, and consider both min–max and min–max regret robustness (see Kasperski & Zieliński, 2016).

This paper is structured as follows. In Section 2, we briefly formalize the setting of variable-sized uncertainty. We then introduce our new compromise approach for min–max regret robustness in Section 3. We present complexity results for the selection problem, the minimum spanning tree problem, and the shortest path problem in Section 4, before discussing the case of min–max robustness in Section 3. In Section 6, we evaluate our approach in a computation experiment, and conclude this paper in Section 7.

2. Variable-sized uncertainty

In the following, we use the notation $[n] := \{1, \ldots, n\}$ and write vectors and matrices in bold, e.g., $\boldsymbol{x} = (x_i)_{i\in[n]}$. We briefly summarize the setting of Chassein and Goerigk (2018), where the term "variable-sized uncertainty" was coined. Consider an uncertain combinatorial problem of the form

$$\min \{\boldsymbol{cx} : \boldsymbol{x} \in \mathcal{X}\} \tag{P(c)}$$

with $\mathcal{X} \subseteq \{0,1\}^n$, and an uncertainty set $\mathcal{U}(\lambda) \subseteq \mathbb{R}^n_+$ that is parameterized by some value $\lambda \in \Lambda$. For example,

- interval-based uncertainty $\mathcal{U}(\lambda) = \prod_{i\in[n]}[(1-\lambda)\hat{c}_i, (1+\lambda)\hat{c}_i]$ with $\Lambda \subseteq [0,1]$,
- general interval-based uncertainty $\mathcal{U}(\lambda) = \prod_{i\in[n]}[\hat{c}_i - \lambda d_i, \hat{c}_i + \lambda d_i]$ with $\boldsymbol{d} \in \mathbb{R}^n_+$, or
- ellipsoidal uncertainty $\mathcal{U}(\lambda) = \{\boldsymbol{c} : \boldsymbol{c} = \hat{\boldsymbol{c}} + \boldsymbol{C}\boldsymbol{\xi}, \|\boldsymbol{\xi}\|_2 \le \lambda\}$ with $\Lambda \subseteq \mathbb{R}_+$, $\boldsymbol{C} \in \mathbb{R}^{n\times m}$, $\boldsymbol{\xi} \in \mathbb{R}^m$.

We call $\hat{\boldsymbol{c}}$ the *nominal scenario*, and any $\hat{\boldsymbol{x}} \in \mathcal{X}$ that is a minimizer of P($\hat{\boldsymbol{c}}$) a *nominal solution*.

In their setting of variable-sized uncertainty, the aim is to find a minimal set of solutions $\mathcal{S} \subseteq \mathcal{X}$ that contains an optimal solution to each robust problem over all λ. Here, the robust problem is either given by the min–max counterpart

$$\min_{\boldsymbol{x}\in\mathcal{X}} \max_{\boldsymbol{c}\in\mathcal{U}(\lambda)} \boldsymbol{cx}$$

or the min–max regret counterpart

$$\min_{\boldsymbol{x}\in\mathcal{X}} \max_{\boldsymbol{c}\in\mathcal{U}(\lambda)} \left(\boldsymbol{cx} - \min_{\boldsymbol{y}\in\mathcal{X}} \boldsymbol{cy}\right).$$

In the case of min–max robustness, such a set can be found through methods from multi-objective optimization in $\mathcal{O}(|\mathcal{S}| \cdot T)$, where T denotes the complexity of the nominal problem, for many reasonable uncertainty sets. However, $\mathcal{S}$ may be exponentially large.

This setting is related to two other approaches from the optimization literature. The first is fuzzy optimization, which draws on possibility distributions to describe the problem uncertainty. A fuzzy set $\tilde{A}$ consists of a reference set Ω and a membership function $\mu_{\tilde{A}} : \Omega \to [0,1]$. The value of the membership function can be interpreted as the degree of membership of an element in $\tilde{A}$. A λ-cut is then defined as all elements with membership at least λ, i.e., the set $\tilde{A}^\lambda = \{v \in \Omega : \mu_{\tilde{A}}(v) \ge \lambda\}$. The λ-cut with $\lambda = 1$ is called the core of a fuzzy set. For details on possibility theory we refer to Dubois and Prade (1988); its relationship to min–max regret is discussed in Kasperski and Zieliński (2010). In variable-sized uncertainty, one may consider the uncertainty set $\mathcal{U}$ as being fuzzy. The set $\prod_{i\in[n]}[\hat{c}_i, \hat{c}_i]$ is then the core of the uncertainty, and each set $\mathcal{U}(\lambda)$ corresponds to a λ-cut in possibility theory. More general ways to model possibility distributions exist, which may constitute an interesting way to extend variable-sized uncertainty in the future.

The second related approach is parametric optimization. In this setting, one considers a family of optimization problems that are parameterized through some value λ. The general goal is to compute regions where the optimal solution does not change, meaning that all possible problems with respect to λ are solved simultaneously. Variable-sized uncertainty can be seen as a parametric problem, where the parameter defines the uncertainty set. We refer to Witting, Ober-Blöbaum, and Dellnitz (2013) for a discussion of robustness and parametric (multi-objective) optimization.

The drawback of variable-sized uncertainty is that the solution set $\mathcal{S}$ may be of exponential size, which would require some processing of solutions before they can be presented to the decision maker. The idea of our compromise approach is to present only one solution with a good overall performance instead. Furthermore, this solution is not necessarily in $\mathcal{S}$, which means that the previous approach might not be able to find it. We introduce our new approach in the following section.

3. Compromise solutions in the min–max regret model

In this paper, we are interested in finding one single solution that performs well on average over all possible uncertainty sizes $\lambda \in \Lambda$. Recall that in classic min–max regret, one considers the problem

$$\min_{\boldsymbol{x}\in\mathcal{X}} \max_{\boldsymbol{c}\in\mathcal{U}(\lambda)} \boldsymbol{cx} - opt(\boldsymbol{c})$$

with $opt(\boldsymbol{c}) = \min_{\boldsymbol{y}\in\mathcal{X}} \boldsymbol{cy}$. We define the compromise approach to variable-sized uncertainty as the following problem:

$$\min\ val(\boldsymbol{x}) \quad \text{with} \quad val(\boldsymbol{x}) = \int_\Lambda \left(\max_{\boldsymbol{c}\in\mathcal{U}(\lambda)} \boldsymbol{cx} - opt(\boldsymbol{c})\right) d\lambda \tag{CMMR}$$

In the following, we focus our analysis on the classic interval uncertainty sets $\mathcal{U}(\lambda) = \prod_{i\in[n]}[(1-\lambda)\hat{c}_i, (1+\lambda)\hat{c}_i]$. To simplify the presentation, we further assume $\Lambda = [0,1]$. The previous work on variable-sized uncertainty aims at presenting the decision maker with a set of solutions, and assumes that the decision maker will then choose the solution that suits his requirements best. Not presenting a single solution but a set of solution respectively a probability distribution of solutions was also proposed in Mastin, Jaillet, and Chin (2015). There the authors introduce the concept of randomized min–max regret in which the goal is to find a probability distribution over solutions such that the expected maximum regret is minimal. Note that in this concept the uncertainty set size is assumed to be fixed.

In contrast, the concept of compromise min–max regret produces only a single solution that represents a good overall trade-off for all uncertainty set sizes. More precise, the compromise min–max regret solution minimizes the average of the maximum regret over all considered uncertainty set sizes. It can therefore be seen in the tradition of goal programming for multi-objective optimization. As a motivation for our approach, consider the example shown in Fig. 1.