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The price-setting newsvendor problem with nonnegative linear additive demand

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A R T I C L E I N F O

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A B S T R A C T

We analyze the price-sensitive newsvendor problem with non-negative linear additive demand. We show that the problem always has an optimal solution and identify random demand distributions for which a unique optimal solution can be computed by showing that, for those distributions, the expected profit is a quasiconcave function of the retail price.

1. Introduction

The price-setting newsvendor problem has been thoroughly analyzed as shown in the survey of Petruzzi and Dada (1999). For the linear additive demand case considered in this paper, Whitin (1955) and Mills (1959) proposed the early formulations of this problem and characterized its optimal solutions. Ernst (1970) and Young (1978) proved the unimodality of the objective function under certain assumptions on the statistical distribution which were generalized by Petruzzi and Dada (1999). Kocabiyikoglu and Popescu (2011), studied the newsvendor problem with a more general demand function and derived general conditions for unimodality of the objective function. Xu, Chen, and Xu (2010) analyzed the effect of demand uncertainty in a price-setting newsvendor model and Zhao and Atkins (2008) studied an equilibrium extension of the newsvendor problem with multiple competitive newsvendors.

Krishnan (2010) pointed out that a nonnegativity constraint should be imposed on the total demand in the linear additive demand formulation in Petruzzi and Dada (1999) to eliminate the possibility of negative actual demand. Otherwise, the computed optimal price p may be suboptimal and the expected profit is underestimated because the actual demand is "choked off" at high optimal p values. Krishnan (2010) observes that, by assuming nonnegativite linear additive demand, the expected profit function may no longer be quasiconcave in p and the problem becomes less tractable. Farahat and Perakis (2010) also recognized the need to incorporate the nonnegativity constraint on the total demand in their model of multiproduct price competition.

In this paper, we consider the price-setting newsvendor problem with nonnegative linear additive demand. We show that the problem still possesses an optimal solution (possibly nonunique) and investigate when the expected profit function is quasiconcave in p so that a unique optimal solution can be computed. We show that this is possible when the expected actual sales can be expressed in closed form and identify a number of random demand distributions for which this is true. We demonstrate our approach through a numerical example.

The rest of the paper is organized as follows. The price-setting newsvendor problem with nonnegative linear additive demand is analyzed in Section 2. Section 3 summarizes the conclusions of this research.

2. The price-setting newsvendor problem with nonnegative linear additive demand

Let $D = d(p) + x$ denote the total linear additive demand, where $d(p) = a - bp$, $a > 0$, $b > 0$, is the price-dependent deterministic demand, p is the retail price and x is the random demand. We denote by q the order quantity and by c the unit cost. This linear deterministic demand function is in effect when consumers decide whether to purchase the product by maximizing the utility function $U(q) = \frac{a}{b}q - \frac{1}{2b}q^2 - pq$ subject to $p \le \frac{a}{b}$ because in that case $q^* = a - bp \ge 0$ is the unique optimal solution of $\max_{q\ge 0} U(q)$. On the other hand, the non-linear deterministic demand function $d(p) = (a - bp)^+$ related to the analysis in this paper is in effect when the consumer utility function $U(q) = \frac{a}{b}q - \frac{1}{2b}q^2 - pq$ is

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maximized with unbounded $p \ge 0$. Observe that when $p > \frac{a}{b}$, $q^* = 0$ is the unique optimal solution of $\max_{q \ge 0} U(q)$.

In the case of a general additive demand function $d(p)$, where $d'(p) < 0$ for $p \ge 0$ and $d(p) < 0$ for large enough p, the utility function $U(q)$ generalizes to $U(q) = \int_0^q d^{-1}(u)du - pq$ subject to $d(p) \ge 0$, where $d^{-1}(u)$ is the inverse demand function. The optimal solution of the $\max_{q \ge 0} U(q)$ problem is $q^* = d(p) \ge 0$. The imposition of a nonnegativity constraint on the total demand leads to a significantly more difficult problem compared to the linear additive case. For this reason, we focus on the linear demand function $d(p) = a - bp$ in the sequel.

The random demand x, has cdf $F(x)$, $\bar{F}(x) = 1 - F(x)$, and pdf $f(x)$ with support on $[0, \infty)$ so that $f(x) = 0$ for $x < 0$ and $f(x) > 0$ for $x \ge 0$. We assume that $f(x)$ is continuously differentiable for $x > 0$ and that the mean μ is finite. The failure rate is defined as $h(x) = \frac{f(x)}{\bar{F}(x)}$. F is IFR (increasing failure rate) if $h'(x) \ge 0$.

In the absence of a nonnegativity constraint on the total demand the newsvendor solves the expected profit maximization problem

$$\max_{p,q} \Pi(p,q) = pE[\min(q, a - bp + x)] - cq. \tag{1}$$

By solving problem (1), it is possible that $d(p^*) = a - bp^* < 0$, where p^* is the optimal price. Since the random term x can be viewed as the market noise, the actual demand realization $d(p^*) + x$ may be negative for small noise x which implies that there is no demand with adverse market conditions. This is in contrast with most literature which assumes that the newsvendor has nonnegative demand even with the worst market conditions.

By imposing a nonnegativity constraint on the total demand, (1) becomes

$$\max_{p,q} \tilde{\Pi}(p,q) = pE[\min(q, (a - bp + x)^+)] - cq, \tag{2}$$

where $z^+ = \max\{0, z\}$. Since $x \ge 0$, $\tilde{\Pi}(p,q) = \Pi(p,q)$ for $p \le \frac{a}{b}$, $q \ge 0$. However, $\tilde{\Pi}(p,q) > \Pi(p,q)$ for $p > \frac{a}{b}$, $q \ge 0$, because, in that case, $(a - bp + x)^+ = 0 > a - bp + x$ when $0 \le x < bp - a$. Note that since we only require the actual demand realization $a - bp + x$ to be nonnegative, $p > \frac{a}{b}$ is allowed when the random demand x satisfies $0 \le x < bp - a$.

Let $q = a - bp + z \ge 0$, where $z \ge 0$ is the stocking factor and define $S(z) = E[\min(z, x)]$ as the expected actual sales given the quantity z. Then, (1) can be written as

$$\max_{p,z} \Pi(p,z) = p(a - bp) + pS(z) - c(a - bp) - cz. \tag{3}$$

2.1. The nonnegative deterministic demand case

Consider the problem (2) restricted to the feasible set where $d(p) = a - bp \ge 0$ so that $p \le \frac{a}{b}$ and $\tilde{\Pi}(p,q) = \Pi(p,q)$. If we let $q = a - bp + z \ge 0$, this restricted version of problem (2) can be written as

$$\max_{p \le \frac{a}{b}, z} \tilde{\Pi}(p,z) = p(a - bp) + pS(z) - c(a - bp) - cz. \tag{4}$$

Let z_0 be the unique solution of $S(z_0) = a - bc$ and define $H(z) = [a + bc + S(z)]\frac{h(z)}{\bar{F}(z)}$. Proposition 1, stated next, specifies when the optimal solution restricted to the feasible set where $d(p) = a - bp \ge 0$ lies in the interior of that feasible set (with $p^* < \frac{a}{b}$) or on its boundary (with $p^* = \frac{a}{b}$).

Proposition 1. *Assume that F is IFR and $a > bc$. Then, the problem $\max_{p \le \frac{a}{b}, z} \Pi(p,z)$ has a unique optimal solution (p^*, z^*), where*

$$p^* = p(z^*) = \begin{cases} \frac{1}{2b}[a + bc + S(z^*)] \in (c, \frac{a}{b}), \text{ if } H(z_0) > 1, \\ \quad \bar{F}(z_0) < \frac{bc}{a}, \\ \frac{a}{b}, \text{ if } H(z_0) \le 1 \text{ or } H(z_0) > 1, \bar{F}(z_0) \ge \frac{bc}{a}, \end{cases}$$

$$z^* = \begin{cases} \textit{the unique solution of } p(z)\bar{F}(z) - c \\ \quad = 0 \ (z^* < z_0), \text{ if } H(z_0) > 1, \bar{F}(z_0) < \dfrac{bc}{a}, \\ \bar{F}^{-1}(\dfrac{bc}{a}) \ (z^* \ge z_0), \text{ if } H(z_0) \le 1 \text{ or } H(z_0) > 1, \bar{F}(z_0) \ge \dfrac{bc}{a}, \end{cases}$$

All proof are relegated to the Appendix.

2.2. The nonpositive deterministic demand case

Consider the problem (2) restricted to the feasible set where $d(p) = a - bp \le 0$, that is $p \ge \frac{a}{b}$. Then, if we let $q = a - bp + z \ge 0$, which is equivalent to $z \ge bp - a$, this restricted version of problem (2) can be written as

$$\max_{p \ge \frac{a}{b}, z \ge bp - a} \tilde{\Pi}(p,z) = pE[\min(a - bp + z, (a - bp + x)^+)] \\ - c(a - bp) - cz. \tag{5}$$

Since for $p \ge \frac{a}{b}$, $\min[a - bp + z, (a - bp + x)^+] = 0$ for $x \le bp - a$ and $\min[a - bp + z, (a - bp + x)^+] = a - bp + \min(z, x)$ for $x \ge bp - a$; then, $E\{\min[a - bp + z, (a - bp + x)^+]\} = S(z) - S(bp - a)$. Thus, (5) can be written as

$$\max_{p \ge \frac{a}{b}, z \ge bp - a} \tilde{\Pi}(p,z) = -pS(bp - a) + pS(z) - c(a - bp) - cz. \tag{6}$$

In view of (3), for $p > \frac{a}{b}$, $z \ge bp - a$, $\tilde{\Pi}(p,z) - \Pi(p,z) = p[(bp - a) - S(bp - a)] > 0$. Let us replace the variable z in (6) by $q = a - bp + z \ge 0$. Then, (6) can be written as

$$\max_{p \ge \frac{a}{b}, q \ge 0} \tilde{\Pi}(p,q) = pS(bp - a + q) - pS(bp - a) - cq. \tag{7}$$

The next proposition shows that the profit function $\tilde{\Pi}(p,q)$ in (7) is bounded and thus (7) always has an optimal (possibly nonunique) solution.

Proposition 2. *Assume that F is IFR and $a > bc$. Then, (7) always has an optimal solution. A unique optimal solution can be computed if we can show that (7) is quasiconcave in p. A critical step in that process is the ability to express $S(z)$ in a closed form. A list of IFR distributions where this is possible includes the Gamma distribution with integer $k \ge 1$ ($f(x) = \frac{1}{(k-1)!}x^{k-1}e^{-x}$), the χ^2 distribution with integer $\frac{\nu}{2} \ge 1$ ($f(x) = \frac{1}{2^{\nu/2}(\nu/2-1)!}x^{\nu/2-1}e^{-x/2}$) and the Chi distribution with integer $k \ge 1$ ($f(x) = \frac{1}{2^{k/2-1}(k-1)!}x^{k-1}e^{-x/2}$), all with support $[0, \infty)$, as well as the uniform distribution $U[0, b]$ and the Power distribution with support $[0, 1]$ ($f(x) = kx^{k-1}$). In contrast, a closed form expression for $S(x)$ is not possible for the Normal distribution, the Logistic distribution and the Weibull distribution.*

In the next proposition, we demonstrate this approach for the gamma distribution with $k = 2$.

Proposition 3. *Assume that $a > bc$. Then, for the Gamma distribution with $k = 2$, where $f(x) = xe^{-x}$, $\tilde{\Pi}(p,q)$ is quasiconcave in p on $[\frac{a}{b}, \infty)$ for any fixed $q \ge 0$. Moreover, the unique optimal solution of $\max_{p \ge \frac{a}{b}} \tilde{\Pi}(p,q)$, is given by*

$$p(q) = \begin{cases} p(q) = \frac{1}{2\nu_1(q)b}\,[\,\nu_1(q)a + \nu_2(q) \\ \quad + \sqrt{4[\nu_1(q)]^2 + [(2 - a)\nu_1(q) - \nu_2(q)]^2}\,], \text{ if } p(q) \ge \dfrac{a}{b}, \\ \dfrac{a}{b}, \text{ if } p(q) < \dfrac{a}{b}, \end{cases}$$

where $\nu_1(q) = 1 - e^{-q}$, $\nu_2(q) = qe^{-q}$. Thus, (7) is equivalent to the single-variable problem $\max_{q \ge 0} \tilde{\Pi}(p(q), q)$ which has an optimal solution.

Proposition 3 does not yield a closed form optimal solution for (7). By emulating the proof of Proposition 3, we obtain the following closed form solution to (7) for the exponential distribution $exp(1)$ with $a > bc$.

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