European Journal of Operational Research 000 (2018) 1–13

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Discrete Optimization

The quadratic shortest path problem: complexity, approximability, and solution methods

Borzou Rostami[a,*], André Chassein[b], Michael Hopf[b], Davide Frey[d], Christoph Buchheim[c], Federico Malucelli[e], Marc Goerigk[f]

[a] CERC-Data Science for Real-time Decision-making, Polytechnique Montréal, GERAD and CIRRELT, Canada
[b] Fachbereich Mathematik, TU Kaiserslautern, Germany
[c] Fakultät für Mathematik, TU Dortmund, Germany
[d] INRIA-Rennes Bretagne Atlantique, Rennes, France
[e] Dipartimento di Elettronica e Informazione, Politecnico di Milano, Italy
[f] Department of Management Science, Lancaster University, United Kingdom

A R T I C L E I N F O

Article history:
Received 13 February 2017
Accepted 30 January 2018
Available online xxx

Keywords:
Combinatorial optimization
Shortest path problem
Quadratic 0–1 optimization
Computational complexity
Branch-and-Bound

A B S T R A C T

We consider the problem of finding a shortest path in a directed graph with a quadratic objective function (the QSPP). We show that the QSPP cannot be approximated unless P = NP. For the case of a convex objective function, an n-approximation algorithm is presented, where n is the number of nodes in the graph, and APX-hardness is shown. Furthermore, we prove that even if only adjacent arcs play a part in the quadratic objective function, the problem still cannot be approximated unless P = NP. In order to solve the problem we first propose a mixed integer programming formulation, and then devise an efficient exact Branch-and-Bound algorithm for the general QSPP, where lower bounds are computed by considering a reformulation scheme that is solvable through a number of minimum cost flow problems. In our computational experiments we solve to optimality different classes of instances with up to 1000 nodes.

1. Introduction

The Shortest Path Problem (SPP) of finding a path in a directed graph from an origin node s to a target node t with minimal arc length is a well-studied combinatorial optimization problem. Many classical algorithms such as Dijkstra's labeling algorithm (Dijkstra, 1959) have been developed to solve the SPP efficiently.

Several extensions of the basic SPP exist to model more complex settings. These include problems where the travel costs of an arc follow a distribution and the shortest path is constrained by parameters such as the variance of the cost of the path (Sivakumar & Batta, 1994), and problems in which additional costs arise from pairs of arcs in a solution (Amaldi, Galbiati, & Maffioli, 2011; Gourvès, Lyra, Martinhon, & Monnot, 2010).

In this paper, we consider the shortest path problem with a quadratic objective function (the QSPP). Specifically, writing the linear objective function of the classical shortest path problem as $c^\top x$ with a cost vector c, the objective function of the QSPP is $x^\top Qx + c^\top x$ with a quadratic matrix Q.

1.1. Applications and related work

One variant of the SPP studied in the literature that is directly related to QSPP is that of finding a variance-constrained shortest path (Sivakumar & Batta, 1994) where the arc costs are not deterministic but follow a distribution and the objective is to find a path with minimum expected costs subject to the constraint that the variance of the costs is less than a specific threshold. In particular, a solution consists of a path that must have both a short expected length and a low risk of exploding costs in an unfortunate event. An application for this problem is the transportation of hazardous materials. Possible approaches to solve the Variance-Constrained Shortest Path problem involve a relaxation in which the quadratic variance constraint is incorporated into the objective function, thus yielding a QSPP problem. In this case, the quadratic part of the objective function is determined by the covariance matrix of the coefficient's probability distributions, and hence convex. In a similar way, instead of bounding the variance, one may search for a solution that considers both the expected cost and the variance of a path as optimization criteria. In Suvrajeet, Pillai, Joshi, and Rathi (2001), the authors consider this as a multi-objective optimization problem. They solve this problem by combining the

* Corresponding author.
E-mail address: borzou.rostami@polimi.it (B. Rostami).

https://doi.org/10.1016/j.ejor.2018.01.054
0377-2217/

Table 1
Our complexity results for different variants of the Quadratic Shortest Path Problem. The entries marked with in asterisk (*) hold true unless NP = P.

PROBLEM	GRAPH TYPE General	Acyclic	Series-parallel graph
QSPP	Not approximable*	Not approximable*	Not approximable*
convex QSPP	APX-hard	APX-hard	APX-hard
AQSPP	Not approximable*	P	P

linear and quadratic objective functions into a single QSPP. Also related to variance-constrained shortest path problems are the so-called reliable shortest paths, see (Chen et al., 2012).

A different type of applications arises from research on network protocols. In Murakami and Kim (1997), the authors study different restoration schemes for self-healing ATM networks. In particular, the authors examine line and end-to-end restoration schemes. In the former, link failures are addressed by routing traffic around the failed link, in the latter, traffic is rerouted by computing an alternative path between source and target. Within their analysis, the authors point out the need to solve a QSPP to address rerouting in the latter scheme. Nevertheless, they do not provide details about the algorithm used to obtain a QSPP solution.

All problems described above involve variants of the classical shortest path problem in which additional costs arise with the presence of pairs of arcs in the solution. Such a setting can be modeled by a quadratic objective function on binary variables associated with each arc, and leads to the definition of a QSPP.

To the best of our knowledge there is no specific method in the literature to solve the QSPP. The only algorithmic approach that has been applied to solve instances of the the QSPP is the one proposed in Buchheim and Traversi (2015). They studied a generic framework for solving binary quadratic programming problems. In their computational experiments, they solve some special classes of quadratic $0-1$ problems including the QSPP.

1.2. Main contributions

In this paper, we analyze the complexity of the general QSPP and several of its special cases. In particular, we show that the general QSPP cannot be approximated unless P = NP. This is done by reducing an instance of the Path with Forbidden Pairs Problem (known to be NP-complete) to a corresponding instance of the QSPP. We also show that, even if we restrict the quadratic part of the cost function to pairs of arcs which are adjacent (AQSPP), the problem still cannot be approximated unless P = NP. This is done by a gap-producing reduction from an instance of 3SAT to an instance of the AQSPP. Moreover, for the convex QSPP where the quadratic form is positive semidefinite and, thus, the objective function is convex, we show that the problem is APX-hard and provide an n-approximation algorithm, where n is number of nodes in the graph. Our complexity results are summarized in Table 1.

From the practical point of view, we present a mixed integer programming formulation whose size is linear in terms of the number of variables in the original quadratic formulation. We also propose an exact Branch-and-Bound algorithm for the general QSPP, where lower bounds are computed by considering a reformulation scheme that is solvable through a number of minimum cost flow problems. In our computational experiments we solve to optimality different types of instances with up to 1000 nodes and show that our results outperform a state-of-the-art solver.

Parts of this paper have been published as conference proceedings (Rostami, Malucelli, Frey, & Buchheim, 2015), where the authors show the NP-hardness of the general QSPP, analyze polynomially solvable special cases, and propose some bounding procedures for the general QSPP.

2. Problem formulation

Let a directed graph $G=(V,A)$ be given, with a source node $s\in V$, a target node $t\in V$, a cost function $c: A\to\mathbb{R}^+$, which maps every arc to a non-negative cost, and a cost function $q: A\times A\to\mathbb{R}^+$ that maps every pair of arcs to a non-negative cost. We denote by $\delta^-(i)=\{j\in V \mid (j,i)\in A\}$ and $\delta^+(i)=\{j\in V \mid (i,j)\in A\}$ the sets of predecessor and successor nodes for any given $i\in V$, by n the number of nodes, and by m the number of arcs. Using binary variables x_{ij} indicating the presence of arc $(i,j)\in A$ on the optimal path, the QSPP is represented as:

$$\text{QSPP:}\quad z^* = \min \sum_{(i,j),(k,l)\in A} q_{ijkl}x_{ij}x_{kl} + \sum_{(i,j)\in A} c_{ij}x_{ij}$$
$$\text{s.t.}\quad x\in X_{st},\ x \text{ binary.} \qquad (1)$$

Here, the feasible region X_{st} is the path polyhedron

$$X_{st} = \left\{0\le x\le 1:\ \sum_{j\in\delta^+(i)} x_{ij} - \sum_{j\in\delta^-(i)} x_{ji} = b(i)\quad \forall i\in V\right\}$$

with $b(i)=1$ for $i=s$, $b(i)=-1$ for $i=t$, and $b(i)=0$ for $i\in V\backslash\{s,t\}$. Note that, like in the case of classic shortest path problems, it is not necessary to include cycle-elimination constraints, as all costs are positive.

Note that the objective function of the QSPP can be represented by a quadratic and a linear term $f(x):=x^TQx+c^Tx$ for an appropriate matrix Q. We can assume without loss of generality that the matrix Q is symmetric and denote the special case where Q is positive semi-definite, i.e., when f is convex, as the convex QSPP.

Next we define some special cases of the QSPP where the quadratic part of the cost function has a *local* structure, meaning that each pair of variables appearing jointly in a quadratic term in the objective function corresponds to a pair of arcs lying close to each other. We define the Adjacent QSPP (AQSPP), where interaction costs of all non-adjacent pair of arcs are assumed to be zero. Therefore, only the quadratic terms of the form $x_{ij}x_{kl}$ with $j=k$ and $i\neq l$ or with $j\neq k$ and $i=l$ have nonzero objective function coefficients.

As a variant of the AQSPP, we may count additional costs for adjacent arc pairs only if these arcs are traversed consecutively. This problem was investigated in Amaldi et al. (2011), Rostami et al. (2015), Gourvès et al. (2010). To distinguish it from the AQSPP, we call it Consecutive QSPP (CQSPP) here. In fact, the AQSPP and the CQSPP are identical if the given graph is acylic. However, for general graphs they are not equivalent. In fact, while the AQSPP is not even approximable in general, as shown in this paper, the CQSPP turns out to be tractable for any graph. This even remains true when taking all arc pairs into account that appear with a fixed maximal distance on the path (Rostami et al., 2015).

3. Complexity results

3.1. The general QSPP

We start our complexity analysis with the observation that the QSPP can be seen as a generalization of the Path with Forbidden Pairs Problem (PFPP). An instance of the PFPP consists of a graph $G=(V,A)$, two nodes $s,t\in V$ and a list of forbidden arc pairs $\mathcal{L}=\{(a_1,\bar{a}_1),\ldots,(a_k,\bar{a}_k)\}$. The goal is to find a path from s to t that contains at most one arc of each arc pair in $\mathcal{L}$. (The problem may also be defined with a list of forbidden vertex pairs). It is known that this problem is NP-complete (Gabow, Maheshwari, & Osterweil, 1976). Every PFPP can be transformed to an equivalent QSPP, which leads to the following theorem.

Theorem 3.1. *The QSPP cannot be approximated unless* P = NP.