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Continuous Optimization

Packing of concave polyhedra with continuous rotations using nonlinear optimisation

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ABSTRACT

We study the problem of packing a given collection of arbitrary, in general concave, polyhedra into a cuboid of minimal volume. Continuous rotations and translations of polyhedra are allowed. In addition, minimal allowable distances between polyhedra are taken into account. We derive an exact mathematical model using adjusted radical free quasi phi-functions for concave polyhedra to describe non-overlapping and distance constraints. The model is a nonlinear programming formulation. We develop an efficient solution algorithm, which employs a fast starting point algorithm and a new compaction procedure. The procedure reduces our problem to a sequence of nonlinear programming subproblems of considerably smaller dimension and a smaller number of nonlinear inequalities. The benefit of this approach is borne out by the computational results, which include a comparison with previously published instances and new instances.

1. Introduction

Cutting and packing problems have a long history of being tackled by the Operational Research community. Where the objects have arbitrary shape, this research has a strong link with the field of computational geometry (see, e.g., Wascher, Hauner, & Schumann, 2007; Bennell & Oliveira, 2008; Miguel, 2014). These problems have a wide spectrum of applications, for example in modern biology, mineralogy, medicine, materials science, nanotechnology, robotics, pattern recognition systems, control systems, space apparatus control systems, as well as in the chemical industry, power engineering, mechanical engineering, shipbuilding, aircraft construction and civil engineering.

At present, the interest in finding effective solutions for packing problems is growing rapidly. This is due to a large number of applications and the development of new and sophisticated methods that can exploit the ever increasing speed of computer processing.

In this paper, we consider the practical problem of packing a collection of non-identical, and in general, concave polyhedra into a cuboid of minimal sizes (in particular volume). We will refer to the problem as the polyhedron packing problem.

An interesting example of applications of the polyhedron packing arises in engineering design. Optimal packing of electronic components and payload has always been a pivotal concern in vehicle engineering, in particular in applications where volume is at a premium, for example embedding avionics in aircraft. The aim is to design an external envelope and determine the configuration of the payload subject to a fixed volume constraint. Alternatively, the approach may be to design an envelope around a fixed packing of the payload and the avionics in order to minimise volume while satisfying a set of mechanical, technical and manoeuvrability constraints.

Another application arises in the recent advent of additive manufacturing (AM), often referred to as 3D printing. There are a variety of different AM technologies that build up objects by adding one very thin layer of material at a time, for example through material extrusion or sintering layers of powder material. This procedure is very slow and not appropriate for repetitive manufacturing but useful for individual items and prototyping. Combining objects into one compact print pattern can reduce the print time, improving capacity utilisation, and reduce the need for extra supporting material that is often required as part of the printing process when objects are arranged in certain configurations.

The polyhedron problems are NP-hard (Chazelle, Edelsbrunner, & Guibas, 1989) and, as a result, solution methodologies generally employ heuristics, for example see Chen, Klotsa, Engel, Damasceno, and Glotzer (2014), Galrão, Oliveira, Gonçalves, and Lopes (2016), Korte and Brouwers (2013), Li, Zhao, Lu, and Xie (2010), Smeets, Odenthal, Vanmaercke, and Ramon (2015), Stroeven and He (2013) and Tasios, Gantapara, and Dijkstra (2014). Some

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researchers develop approaches based on mathematical modelling and general optimisation procedures; for example see Egeblad, Nielsen, and Brazil (2009), Fasano (2013) and Torquato and Jiao (2009).

Egeblad et al. (2009) present an efficient solution method for packing polyhedra within the bounds of a container (a polyhedron). The central geometric operation of the method is an exact horizontal or vertical translation of a given polyhedron to a position, which minimises its volume of overlap with all other polyhedra. The translation algorithm is embedded into a local search heuristic. Additional details are given for the three-dimensional case and appropriate results are reported for the problem of packing polyhedra into a rectangular parallelepiped. Utilisation of container space is improved by an average of more than 14 percentage points compared to previous methods proposed in Stoyan, Gil, Scheithauer, Pankratov, and Magdalina (2005). In the experiments the largest total volume of overlap allowed in a solution corresponds to 0.01% of the total volume of all polyhedra for the given problem.

Liu, Liu, Cao, and Yao (2015) propose a new constructive algorithm, called HAPE3D, which is a heuristic algorithm based on the principle of minimum total "potential energy" for the 3D irregular packing problem, involving packing a set of irregularly shaped polyhedrons into a box-shaped container with fixed width and length but unconstrained height. The objective is to allocate all the polyhedrons in the container, and thus minimise the waste or maximise profit. HAPE3D can deal with arbitrarily shaped polyhedrons, which can be rotated around each coordinate axis at different angles. The most outstanding merit is that HAPE3D does not need to calculate no-fit polyhedrons. HAPE3D can also be hybridised with a meta-heuristic algorithm such as simulated annealing. Two groups of computational experiments demonstrate the good performance of HAPE3D and prove that it can be hybridised with a meta-heuristic algorithm that further improves the packing quality.

Our approach is based on the mathematical modelling of relations between geometric objects and allowing the packing problem to be formulated as a nonlinear programming problem. To this end we use the phi-function technique (see, Chernov, Stoyan, & Romanova, 2010) to provide an analytic description of objects placed in a container taking into account their *continuous rotations* and *translations*. At present phi-functions for the simplest 3D-objects, such as parallelepipeds, convex polyhedra and spheres are considered in Stoyan and Chugay (2012). Phi-functions for 3D-objects, in particular polyhedra, can be highly complicated analytically, since they involve many radicals and maximum operators, and are therefore difficult for NLP-solvers to solve.

In this paper we apply the *quasi phi-functions* concept introduced in Stoyan, Pankratov, and Romanova (2016), which is based on the idea proposed by Kallrath (2009) to use a separating plane to model non-overlapping constraints for circles and convex polygons. The concept of *quasi phi-functions* extends the domain of *phi-functions* by including auxiliary variables. The new functions can be described by analytical formulas that are substantially simpler than those used for phi-functions, for some types of objects, in particular, for convex polyhedra.

The use of quasi phi-functions, instead of phi-functions, allows us to describe (or simplify) the non-overlapping constraints. While this makes our models easier to solve, this comes at a price, which is performing the optimisation over a larger set of parameters, including the extra (auxiliary) variables used by the quasi phi-functions. Our approach is capable of finding a good local optimal solution in reasonable computational time.

The phi- and quasi phi-functions have been widely and successfully used to model a variety of packing problems, as in Chernov et al. (2010), Pankratov, Romanova, and Chugay (2015), Stoyan, Gil, and Pankratov (2004) and Stoyan et al. (2005, 2016). In the current manuscript, we consider packing problem of *concave* polyhedra. The contributions of the work presented in this manuscript are as follows:

- We construct radical free *quasi phi-functions* to describe analytically the non-overlapping constraints for *concave* polyhedra and *adjusted quasi phi-functions* to describe analytically the minimal allowable distances between *concave* polyhedra.
- We derive an *exact mathematical model* of the optimal packing problem of *concave* polyhedra as a *continuous nonlinear programming problem*. Our feasible region is described by a system of inequalities with infinitely differentiable functions.
- We develop an efficient solution algorithm, which employs a clear and simple starting point algorithm and a new and original optimisation procedure (called COMPOLY) for the compaction of concave polyhedra. The COMPOLY procedure reduces our problem to a sequence of NLP subproblems of considerably smaller dimension and a smaller number of nonlinear inequalities. The procedure allows us to search for local optimal solutions of the packing problem.
- Our approach allows us to apply state of the art NLP solvers to the optimal packing problem of *concave* polyhedra.

The paper is organised as follows: in Section 2 we formulate the polyhedron packing problem. In Section 3 we give definitions of a phi-function and a quasi phi-function, an adjusted phi-function and an adjusted quasi phi-function and derive related functions for an analytical description of non-overlapping, containment and distance constraints in the problem. In Section 4 we provide an exact mathematical model in the form of a nonlinear programming problem by means of the phi-function technique. In Section 5 we describe a solution algorithm, which involves a fast starting point and efficient local optimisation procedures. In Section 6 we present our computational results for some new instances and several instances studied before. Finally, Section 7 concludes this paper with a brief summary and a discussion about our future research directions.

2. Problem formulation

We consider here the packing problem in the following setting. Let Ω denote a cuboid, $\Omega = \{(x, y, z) \in R^3 : 0 \le x \le l, 0 \le y \le w, 0 \le z \le h\}$. It should be noted that each of the three dimensions (l or w or h) can be variable. Let $\{1, 2, ..., N\} = J_N$ and a set of polyhedra $\mathbb{Q}_q$, $q \in J_N$ be given.

Each polyhedron $\mathbb{Q}_q$ can be concave or convex. With each polyhedron $\mathbb{Q}_q$ we associate its local coordinate system with origin denoted by v_q.

Assume that each concave polyhedron $\mathbb{Q}_q$ is presented as a union of convex polyhedra K_j^q, $j = 1,\dots,n_q$. With each convex polyhedron K_j^q we associate the local coordinate system of the polyhedron $\mathbb{Q}_q$. Each convex polyhedron K_j^q is defined by its vertices p_s^{qj}, $s = 1, \dots, m_j^q$, in the local coordinate system of $\mathbb{Q}_q$.

We give here input data that form a concave polyhedron $\mathbb{Q}_q$ by two lists:

- List_1 contains the vertex coordinates of all the convex polyhedra K_j^q, $j = 1,\dots,n_q$, and
- List_2 contains the index sets J_j^q, $j = 1,\dots,n_q$, of the numbers of vertices (with respect to List_1) that define appropriate convex polyhedra K_j^q, $j = 1,\dots,n_q$.

We note that List_1 involves all the original vertices of the concave polyhedron and, in general, additional vertices that appear as a result of decomposing the concave polyhedron into convex polyhedra. See Appendix A for details.