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A global tolerance approach to sensitivity analysis in linear programming

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ABSTRACT

This paper takes a fresh look at sensitivity analysis in linear programming. We propose a merged approach that brings together the insights of Wendell's tolerance and Wagner's global sensitivity approaches. The modeler/analyst is then capable of answering questions concerning stability, trend, model structure, and data prioritization simultaneously. Analytical as well as numerical aspects of the approach are discussed for separate as well as simultaneous variations in the objective function coefficients and right-hand side terms. A corresponding efficient numerical implementation procedure is proposed. A classical production problem illustrates the findings.

1. Introduction

Linear programs (LPs) are a central modelling tool for the solution of complex decision problems (Bradley, Hax, & Magnanti, 1977; Dantzig, 1982). Due to the availability of fast-solving algorithms, the use of LPs is now widespread. However, finding an optimal solution is only the beginning. After a model has been formulated and solved, we face the delicate task of testing results and developing managerial insights that guide the implementation of the optimal policy.

The question raised in the literature since the seminal work of Little (1970) is: what are the insights that can be systematically inferred for LPs? We believe that answering this question involves focusing on a number of key properties:

- Trend: is the variation in an uncertain datum going to increase or decrease the optimal profit?
- Presence of interactions: if two data vary simultaneously, is the resulting change the simple direct sum of their individual effects; if not, how relevant are interactions?
- Data prioritization: what datum is more responsible for variations in the optimal profit?
- Data fixing: what data can be fixed, as they have a very low impact on the variation in the optimal profit?
- Stability: is the optimal policy stable to the variations in the data?

Two main approaches to sensitivity analysis in linear programming are the tolerance approach of Wendell (1984, 1985) and the global approach of Wagner (1995). The goal of the tolerance approach is the determination of the maximum percent variation in the data under which the base case optimal solution remains optimal. The goal of Wagner's global sensitivity analysis is the identification of the key variability drivers. Thus, neither a tolerance nor a global sensitivity approach allows the analyst to address the above sensitivity questions simultaneously. In fact, a tolerance analysis (per se) would answer solely the last question. Wagner's sensitivity measures (per se) would address the data prioritization and data fixing questions. Interactions quantification has not been addressed so far in LP applications. Trend identification has been addressed, while not formally, in the so-called ordinary sensitivity analysis (Koltai & Tatay., 2011; Koltai & Terlaky., 2000). However, ordinary sensitivity analysis foresees variation of one datum at a time and has several limitations (Jan, 1997).

Herein we address the research question of merging the global and tolerance sensitivity approaches into a global tolerance approach for a sensitivity analysis of LPs that: (a) allows for simultaneous variations in the data; (b) yields an answer to the five sensitivity questions mentioned above; and (c) is not too computationally demanding.

To merge them, we first show how Wagner's approach can be fruitfully nested in the high dimensional model representation (HDMR) framework (Liu & Owen., 2006; Sobol', 1993b). This

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nesting enables us to bring a series of recent conceptual and computational innovations in HDMR theory to sensitivity analysis in LP. These innovations include: a reinterpretation of Wagnerâs two measures as first and total order effects with respect to an analysis of variance (ANOVA) decomposition of variations in the optimal value of the objective function (henceforth called optimal value function); the introduction of the concept of one-way ANOVA sensitivity functions as a sensitivity tool in LP; and the application of efficient computational HDMR methods for analyzing sensitivity in linear optimization. We discuss numerical and analytical aspects in detail. In particular, we show that numerical values of Wagner's sensitivity measures and the behavior of one-way sensitivity functions are intertwined with stability insights.

Using the notion of an uncertainty set $\mathcal{U}$ (Dantzig, 1955, 1963) as the bridge that links the data variations in Wendell's and Wagner's approaches, the merging enables us to show how the numerical values of Wagner's sensitivity measures and the behavior of one-way sensitivity functions are intertwined with stability insights. In particular, when an uncertainty set $\mathcal{U}$ is given by a hyperbox which in turn is a subset of an optimal coefficient set, then we show that: Wagners measures and one-way sensitivity functions can be characterized analytically; all one-way sensitivity functions are linear; Wagners sensitivity measures sum to unity; and no interactions are present. Conversely, if we register non-null higher order variance-based sensitivity measures, slope changes or curvature in one-way sensitivity functions, then we are informed that $\mathcal{U}$ intersects multiple optimal coefficient sets and the base case optimal policy is not stable. When $\mathcal{U}$ intersects multiple coefficient sets, we show that interactions emerge due to the piecewise linear nature of the optimal value function and not because of the presence of multiplicative terms in the optimal value function. This result is peculiar to LPs. Our analysis is not limited to variations in the objective function coefficients, but we give corresponding results for variations in right-hand-side (RHS) terms and in joint variations of the coefficients and RHS terms.

We rely on the sparse grid interpolation method of Buzzard (2012) that permits us to estimate all relevant sensitivity measures with a parsimonious number of the LP model evaluation. To assess whether it is possible to obtain sensitivity measures numerically within reasonable computational times, we perform a series of tests with problems from the Netlib database. We use the classic production problem from Nahmias and Olsen. (2015) to illustrate the approach and discuss the insights obtained from our results as they apply to each of the five key properties described above.

The remainder of the paper is organized as follows. Section 2 reviews the essentials of tolerance and global sensitivity analysis. Section 3 takes a fresh look at Wagner's variance based sensitivity measures through the HDMR framework. Then Section 4 links the tolerance approach with the HDMR framework and gives results for variations in the objective function coefficients (Section 4.1), the RHS terms (Section 4.2), and simultaneous RHS and coefficients variations (Section 4.3). Section 5 evaluates the computational feasibility of the approach using a sample of Netlib problems and applies the findings to the Nahamias test case. Section 6 concludes the work and proposes future research perspectives.

2. Tolerance sensitivity: A review

Consider a LP in standard form

$$\max_{\mathbf{x}} \mathbf{cx}$$
$$s.t.$$
$$A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0, \tag{1}$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ is the vector of objective function coefficients, A is the coefficient matrix, and $\mathbf{b}$ is the vector of RHS terms. The linear objective function $y = \mathbf{cx}$ defines the value of profit (or whatever payoff is of interest to the decision maker). We call a vector that solves the problem in (1) an optimal solution and we denote it by $\mathbf{x}^*$, and we call the corresponding value of the objective function the optimal profit (denoted by y^*). Both the optimal solution and the optimal profit depend on the values assigned to A, $\mathbf{b}$, and $\mathbf{c}$. If the analyst specifies a base case value $\widehat{A}, \widehat{\mathbf{b}}, \widehat{\mathbf{c}}$ for the data, then using these in (1) gives the base case optimal solution and optimal profit, which we denote by $\widehat{\mathbf{x}}^*$ and $\widehat{y}^*$, respectively.

Work in the area of sensitivity in LP dates back into the 1950s (see Gal, 1997 for a historical overview) and it is still an active research area (see, among others, Filippi, 2010; Koltai & Tatay., 2011; Shahin, Hanafizadeh, & Hladk, 2016; Xu & Burer., 2016). We also refer to Shahin et al. (2016) and the references therein for a thorough review of the literature. Contemporary LP software implements solely a so-called ordinary sensitivity analysis, whose limitations are well known – see, among others, Gal (1992); Jan (1997); Koltai and Tatay. (2011); Koltai and Terlaky. (2000). As noted by Bradley et al. (1977) and as termed the 100 percent rule by them, ordinary sensitivity analysis can be applied to consider simultaneous variations of the coefficients/terms within the convex hull of the ordinary intervals. However, as explained in Ward and Wendell. (1990); Wendell (1985), the use of the 100 percent rule has significant computational limitations as well as a serious conceptual challenge in requiring a decision-maker to think in terms of additive fractional changes of percent variations.

Following the characterization of tolerance sensitivity (see Wendell, 1984, 1985), we consider the following perturbed form of Problem (1):

$$\max_{\mathbf{x}} \sum_{j=1}^{n} (\widehat{c}_j + \gamma_j c'_j) x_j$$
$$s.t.$$
$$\sum_{j=1}^{n} (\widehat{a}_{i,j} + \alpha_{i,j} a'_{i,j}) x_j = \widehat{b}_i + \delta_i b'_i, \text{ for } i = 1, 2, \ldots, m$$
$$x_j \geq 0, \text{ for } j = 1, 2, \ldots, n, \tag{2}$$

where A', b' and c' denote matrices (vectors) of selected data and where $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{\alpha}$ denote perturbations. If $\mathbf{c}' = \widehat{\mathbf{c}}$, $\mathbf{b}' = \widehat{\mathbf{b}}$, and $A' = \widehat{A}$, then $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$ represent vectors of percentage variations (or errors) in the data around the base case value (Wendell, 1985). In some cases we may be able to specify a region within which the perturbations are known to vary. Such a region is called an a priori set of permissible perturbations (permissible set, henceforth). We denote this a priori set by $\mathcal{P}$ and without ambiguity $\mathcal{P}$ includes whatever perturbations (objective function, right-hand-side, constraint coefficients) are being considered.

In tolerance sensitivity it is assumed that $\widehat{A}$ has full row rank (Wendell, 1985) and that $\widehat{\mathbf{x}}^*$ is an optimal basic feasible solution. Let $J = \{1, 2, \ldots, n\}$ denote the set of all indices, let $J^* = \{j_1, j_2, \ldots, j_r\}$ denote the indices in the optimal basis and let $K = J \backslash J^*$ denote the indices outside the optimal basis. Then, let B^{-1} and $B^{-1}_{i,\cdot}$ denote the inverse of the optimal basis matrix and its ith row, respectively. Consider next variations in the objective function coefficients and let the data vary in a given permissible set of objective function perturbations. Tolerance sensitivity aims at determining the largest value τ, denoted by τ^*, such that if $\boldsymbol{\gamma}$ is in the permissible set and if $-\tau \leq \gamma_j \leq \tau$, then the optimal basis (including the slack variables) is unaltered. The set of γ's within the permissible set and within the intervals $-\tau^* \leq \gamma_j \leq \tau^*$ is called the maximum *tolerance region*, and the number τ^* is called *amaximum tolerance for the coefficients*. As given in Wendell (1984), finding the maximum tolerance can be viewed as a two-step