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Decision Support

The characterization of affine maximizers on restricted domains with two alternatives

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ABSTRACT

In the framework of implementable social choice functions, we present an axiomatic characterization of affine maximizers for an important missing case in the literature: that of two alternatives with restricted domain. We use two independent conditions: Positive Association of Differences and an independence condition.

1. Introduction

The implementability of social choice functions is an important concept for many operations research problems (e.g., Fadaei & Bichler, 2017; Papakonstantinou & Bogetoft, 2017; Thirumulanathan, Vinay, Bhashyam, & Sundaresan, 2017; Kakade, Lobel, & Nazerzadeh, 2013). Roberts (1979) showed that every implementable social choice function satisfies a condition named PAD (Positive Association of Differences). Conversely, when there are at least three alternatives and the domains of individual preferences are unrestricted, he showed that PAD implies that any onto social choice function is an affine maximizer. When there are two alternatives only, it is well-known that Roberts' Theorem does not hold because there exist social choice functions satisfying PAD on unrestricted domains and that are not affine maximizers. In previous work (Marchant & Mishra, 2015), we have shown that an Independence condition must be added to PAD in order to characterize affine maximizers when there are only two alternatives and when the domain of the valuations consist of an open interval unbounded from above.

Yet, in some applications, it is not realistic to suppose that the domain of valuations is unbounded from above. Suppose for example a budget-constrained planner is considering to provide one of two public goods: either open a park or open a football stadium. It is reasonable to assume that both the public goods have positive valuation to agents – thus the valuations have a lower bound. Also, it is natural that the planner can always subsidize enough amount of money to the agents and not provide any of the public goods. In other words, there is also an upper bound on the valuations. That is why, in this paper, we show that the same conditions as in Marchant and Mishra (2015) characterize the affine maximizers with two alternatives and domains of valuations consisting of an open interval, without the unboundedness restriction.

Section 2 is devoted to the definitions and the result. The proof is presented in Section 3.

2. Definitions, axioms and result

Let $M = \{1, \dots, m\}$ be a finite set of agents. The set of outcomes or social states is denoted by $A = \{a, b\}$. Each outcome is valued by each agent. The valuation of outcome a (resp. b) by agent i is drawn from some real open interval L_i. We define $S = \Pi_{i \in M} L_i$. A vector $x \in S$ represents the valuations of an outcome by all agents. In their characterization of affine maximizers, Marchant and Mishra (2015) assumed that L_i is unbounded from above, for each agent i. Since this restriction can be unrealistic in many applications, we will not assume it in this paper.

An allocation rule is a mapping f: $S \times S \to A$: $(x, s) \to f(x, s)$, where x (resp. s) is the vector of valuations of a (resp. b) by all agents.

Vector inequalities: for $x, y \in \mathbb{R}^n$, $x \gg y$ iff $x_i > y_i$ for $i = 1, \dots, n$; $x > y$ iff $x_i \geq y_i$ for $i = 1, \dots, n$ and $x_j > y_j$ for some j; $x \geq y$ iff $x_i \geq y_i$ for $i = 1, \dots, n$.

We now present the conditions that we will need in order to characterize the affine maximizers when there are only two alter-

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natives. The first one is the well-known *Positive Association of Differences* introduced by Roberts (1979).[2]

PAD: for all x, y, s, t, if $x-y \gg s-t$ and $f(y,t)=a$ then $f(x,s)=a$.

Notice that PAD implies the symmetric condition : $s-t \gg x-y$ and $f(y,t)=b$ implies $f(x,s)=b$.

Our next condition is a form of independence. It was first used by Marchant and Mishra (2015).

Independence: For all $s, t, x, y \in S$,

$$\left.\begin{array}{r} f(x,t)=a \\ \text{and} \\ f(y,s)=a \end{array}\right\} \Rightarrow \begin{cases} f(y+\varepsilon,t)=a, & \forall \varepsilon \gg 0 : y+\varepsilon \in S \\ \text{or} \\ f(x+\varepsilon,s)=a, & \forall \varepsilon \gg 0 : x+\varepsilon \in S. \end{cases}$$

The intuition behind this condition is that elements of S can be ordered from "more in favor of a" to "less in favor of a". Indeed, suppose the elements of S can be ordered and suppose the antecedent of Independence is satisfied. Then either x is not less in favor of a than y or y is not less in favor of a than x. In the first case, $f(x+\varepsilon,s)=a$ and in the second case, $f(y+\varepsilon,t)=a$. We do not consider this condition as compelling. Whether it is appealing or not can depend on the context and the social planner. Paraphrasing Sen (1976), p.254, our independence condition is not designed "to provide an axiomatic justification of" affine maximizers. Instead, we chose "a set of axioms with the focus on transparency rather than on immediate appeal" (Sen, 1976, p. 259).[3]

In order to help the reader have a better grasp of the conditions presented so far, we now provide two examples showing that PAD and Independence are logically independent.

Example 1. Let $L_i =]0,100[$ for all $i \in M = \{1,2,3\}$ and define the allocation rule f with three agents as follows: for all $x, s \in S$,

$$f(x,s)=a \iff \sum_{i\in M} x_i^2 > \sum_{i\in M} s_i^2.$$

This allocation rule violates PAD. To see this, use $x=(16,10)$, $y=(5,9)$, $s=(20,2)$ and $t=(10,2)$. This allocation rule satisfies Independence. Indeed, suppose $f(x,t)=a$ and $f(y,s)=a$. This implies $\sum_{i\in M} x_i^2 > \sum_{i\in M} t_i^2$ and $\sum_{i\in M} y_i^2 > \sum_{i\in M} s_i^2$. Two cases are possible.

- $\sum_{i\in M} x_i^2 > \sum_{i\in M} y_i^2$. Then $\sum_{i\in M} x_i^2 > \sum_{i\in M} s_i^2$ and $\sum_{i\in M}(x_i+\varepsilon_i)^2 > \sum_{i\in M} s_i^2$ for all $\varepsilon \gg 0$. Hence $f(x+\varepsilon,s)=a$.
- $\sum_{i\in M} x_i^2 \le \sum_{i\in M} y_i^2$. Then $\sum_{i\in M} y_i^2 > \sum_{i\in M} t_i^2$ and $\sum_{i\in M}(y_i+\varepsilon_i)^2 > \sum_{i\in M} t_i^2$ for all $\varepsilon \gg 0$. Hence $f(y+\varepsilon,t)=a$.

So, at least one of $f(y+\varepsilon,t)$ and $f(x+\varepsilon,x)$ is equal to a as required by Independence.

Example 2. Let $L_i =]0,10[$ for all $i \in M = \{1,2,3\}$ and define the allocation rule f with three agents as follows:

$$f(x,s)=a \iff \sum_{i\in M}(x_i-s_i)^3 \ge 0.$$

This allocation rule violates Independence and satisfies PAD. To check that it violates Independence, use $x=(6,3,7)$, $y=(6,7,1)$, $t=(9,6,3)$ and $s=(9,1,6)$. We have $f(x,t)=a$, $f(y,s)=a$, $f(y+\varepsilon,t)=b$ and $f(x+\varepsilon,s)=b$ with $\varepsilon=(1/10,1/10,1/10)$. We now prove that it satisfies PAD. Suppose $f(y,t)=a$ and $x-y \gg s-t$. Then $\sum_{i\in M}(y_i-t_i)^3 \ge 0$ and $x-s \gg y-t$ (or $x_i-s_i > y_i-t_i$ for $i\in M$). This implies $\sum_{i\in M}(x_i-s_i)^3 > \sum_{i\in M}(y_i-t_i)^3 \ge 0$ because the third power is strictly monotonic. Hence $f(x,s)=a$.

We are now ready to state our result.

[2] Roberts shows PAD is implied by an incentive compatibility condition.

[3] Our view of the axiomatic analysis is also close in spirit to that of Thomson (2001), in a different domain.

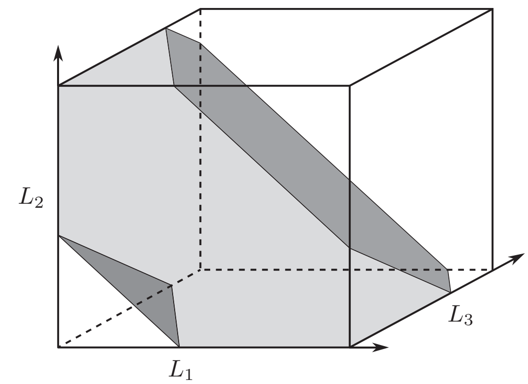

Fig. 1. An affine maximizer, with three agents with valuation domains L_1, L_2 and L_3. The set D is the grey zone between the two darker planes. Under D, b is chosen everywhere and above D, a is chosen everywhere.

Theorem 1. *Suppose for every $i \in M$, L_i is an open interval. The allocation rule f satisfies PAD and Independence iff there is $\lambda \in \mathbb{R}^M$ with $\lambda > 0$ and a real-valued mapping $\kappa : A \to \mathbb{R}$ such that, for all $x, s \in S$,*

$$\sum_{i\in M} \lambda_i x_i + \kappa(a) > \sum_{i\in M} \lambda_i s_i + \kappa(b) \Rightarrow f(x,s)=a$$
$$\sum_{i\in M} \lambda_i x_i + \kappa(a) < \sum_{i\in M} \lambda_i s_i + \kappa(b) \Rightarrow f(x,s)=b.$$

This result is essentially identical to Theorem 2 in Marchant and Mishra (2015), but without the unboundedness restriction. The proof technique used here is different from the one in Marchant and Mishra (2015). The reason why we can now prove a stronger result using the same conditions as in Marchant and Mishra (2015) is perhaps the different technique, but it is perhaps merely the fact that we worked hard to go around all technical problems raised by the boundaries.

Since affine maximizers and our two conditions have been extensively discussed elsewhere, we do not discuss them and we merely present the proof of Theorem 1.

3. Proof

An allocation rule is single-valued and, formally, it is therefore never the case that a and b tie. Yet, if $f(x+\varepsilon,t)=a$ and $f(x-\varepsilon,t)=b$ for all $\varepsilon \gg 0$,[4] we can consider that the valuation x exactly offsets t: any slight change in favor of a (or b) immediately results in a winning (or b). This will be denoted by $x\,T\,t$.

Define $D=\{x \in S : x\,T\,t \text{ for some } t \in S\}$. The set D is the set of all valuations of a that can be offset by some valuation t of b. It is represented in Fig. 1 in the case of an affine maximizer with three agents. If $\kappa(a)=\kappa(b)$, then D is the whole box S. The larger the absolute difference $|\kappa(a)-\kappa(b)|$, the smaller D. If the absolute difference is very large, then D is empty.

Define the relation $\succsim$ on D by $x \succsim y$ iff, for all $t \in S$, we have $f(y,t)=a \Rightarrow [f(x+\varepsilon,t)=a,\ \forall \varepsilon \gg 0]$. When it is not the case that $x \succsim y$, we write $x \not\succsim y$. The binary relation $\succsim$ will play a central role in the proof. Fig. 2 depicts an indifference surface of the relation $\succsim$ in the case of an affine maximizer with three agents.

The proof works as follows. First, we will prove that $\succsim$ is a monotonic weak order (Lemmas 1–3). Lemmas 4–7 are a first attempt at understanding the shape of D (the domain of definition of $\succsim$). With Lemma 8, we will prove that $\succsim$ is continuous. Lemmas 10 and 11 teach us that D is connected while Lemmas 12 and

[4] Strictly speaking, we should write "for all $\varepsilon \gg 0$ such that $x+\varepsilon \in S$ and $x-\varepsilon \in S$", because, otherwise, it can happen that $f(x+\varepsilon,t)$ or $f(x-\varepsilon,t)$ is not defined. This would make the paper very cumbersome and we will therefore omit it.