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Innovative Applications of O.R.

Single-transform formulas for pricing Asian options in a general approximation framework under Markov processes

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Recently, Cai, Song, and Kou (2015) proposed closed-form *double* transform approximation formulas for prices of both discretely and continuously monitored Asian options under the setting of a general continuous-time Markov chain. In this note, we analytically invert the $\mathcal{Z}$-transform and the Laplace transform involved in their final results, respectively, for the discretely and the continuously monitored cases, and we obtain explicit *single* Laplace transforms of option prices. This reduction in the dimension of numerical integral has meaningful consequences both in computational efficiency and in practical implementation of the formulas. Extensive numerical experiments illustrate the improved performance of our results.

1. Introduction

Asian options are popular path-dependent options actively traded in the financial markets, yet their valuation is challenging and has attracted a significant amount of interest in the literature. Even in the case of the Black–Scholes model, there is no exact closed-form solution for the price of the Asian option; thus there have been various methods proposed in the literature. Methods based on partial differential equations (PDE) have been proposed in Rogers and Shi (1995) and Vecer (2001). Monte Carlo simulation methods with control variates have been developed in Fu, Madan, and Wang (1999), Dingeç and Hörmann (2012) and Shiraya and Takahashi (2017), among others. With respect to a series expansion approach, Linetsky (2004) proposed an elegant spectral expansion method to price arithmetic Asian options, and Cai, Li, and Shi (2014) also developed a novel asymptotic expansion approach. Accurate lower and upper bounds to prices of arithmetic Asian options have been recently obtained in Reynaerts, Vanmaele, Dhaene, and Deelstra (2006) and Fusai and Kyriakou (2016). For pricing discretely monitored Asian options, some convex programming based super-replication strategies have been analyzed in Kahalé (2017), and a robust linear optimization technique has been proposed in Bandi and Bertsimas (2014). Moreover, transform-based methods have been popular in the recent literature (see, e.g., Cai & Kou, 2012; Fusai & Meucci, 2008; Fusai, Marena, & Roncoroni, 2008; Sesana, Marazzina, & Fusai, 2014; Kirkby, 2016, among others), where Fourier or Laplace transforms of Asian option prices are characterized.

A common feature of the above-mentioned literature is that these papers usually focus on one specific type of Asian option (discretely monitored or continuously monitored), and a particular class of underlying stochastic dynamics, e.g., geometric Brownian motion, exponential Lévy process, etc. Important progress was recently achieved in Cai, Song, and Kou (2015), in which the authors proposed a general unified framework for pricing both discretely and continuously monitored Asian options under one-dimensional Markov processes through an elegant functional equation approach. They employed results from Mijatović and Pistorius (2013) on a weak approximation scheme from the continuous-time Markov chain (CTMC) to the Markov process, and explicitly solved the functional equations in the CTMC case. The resulting analytical approximate solutions provided in their paper are given as a $\mathcal{Z}$-Laplace transform for the discrete case, and a two-dimensional Laplace transform for the continuous case (Cai et al., 2015, Theorem 2).

In this note, we explicitly invert the $\mathcal{Z}$-transform (discretely monitored case) and the Laplace transform (continuously monitored case) involved in the final results of Cai et al. (2015). We obtain explicit *single* Laplace transforms for both discretely and continuously monitored Asian options. Such new formulas improve upon the existing double transforms in Cai et al. (2015, Theorem 2); the complexity and computational costs are significantly

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reduced. Extensive numerical experiments illustrate the improved performance of our results.

The rest of this note is organized as follows. Section 2 presents our main results as well as some discussion. A set of computational examples is given in Section 3 to illustrate the performance of our new formulas.

2. Main result

Suppose we are given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, assume that $\mathbb{P}$ is the risk-neutral measure and denote by r the risk-free interest rate. Assume that the dividend rate is 0, without loss of generality. One can construct a suitable CTMC to approximate a general one-dimensional Markov process (the stock price); see Mijatović and Pistorius (2013) or Cai et al. (2015, p. 544). Thus in the following, we shall restrict our discussions to CTMCs.

Let $\{X_t\}_{t\geq 0}$ be a non-negative CTMC with finite state space $\{x_1, \dots, x_N\}$, whose transition probability matrix is $\mathbf{P}(t) = (p_{ij}(t))_{N\times N}$, where $p_{ij}(t) = \mathbb{P}(X_{t+u} = x_j \mid X_u = x_i)$, $1 \leq i, j \leq N$, with $t, u \geq 0$. Its transition rate matrix is $\mathbf{G} = (q_{ij})_{N\times N}$, where $q_{ij} = p'_{ij}(0)$, $1 \leq i, j \leq N$. Define $\mathbf{x} = (x_1, \dots, x_N)^T$, and let $\mathbf{I} = \mathbf{I}_{N\times N}$ denote the identity matrix. We use $\mathbf{1} = \mathbf{1}_{N\times 1}$ to denote an $N \times 1$ column vector with all entries equal to 1. Let $\mathbf{D} = (d_{ij})_{N\times N}$ be a diagonal matrix with $d_{jj} = x_j$, $j = 1, \dots, N$. We use $\mathbb{E}^x[\cdot]$ to denote the expectation conditional on $X_0 = x$.

Consider the following payoff functions studied in Eq. (1) (Cai et al., 2015):

$$v_c(t, k; x) = \mathbb{E}^x[(A_t - k)^+], \qquad v_d(n, k; x) = \mathbb{E}^x[(B_n - k)^+],$$

where $A_t := \int_0^t X_u du$ and $B_n := \sum_{i=0}^n X_{t_i}$. Denoting by T the maturity and K the strike price, the price of the continuously monitored Asian call option $V_c(T, K; x)$ at time 0 is given by $(e^{-rT}/T) v_c(T, TK; x)$. Similarly, the price of the discretely monitored Asian call option $V_d(n, K; x)$ at time 0 is given by $(e^{-rT}/(n+1)) v_d(n, (n+1)K; x)$. Henceforth, $\Delta = T/n$. The following is our main result, which improves Theorem 2 of Cai et al. (2015).

Define $v_c(n, k; \mathbf{x}) := (v_c(n, k; x_1), \dots, v_c(n, k; x_N))^T$ and $v_d(n, k; \mathbf{x}) := (v_d(n, k; x_1), \dots, v_d(n, k; x_N))^T$.

Proposition 1 (Single-Laplace transforms for fixed strike Asian options)**.**

(i) *(Discretely monitored Asian options)*
Let $g_d(n, \theta; \mathbf{x}) := \int_0^\infty e^{-\theta k} v_d(n, k; \mathbf{x}) dk$. Then for any complex θ such that $Re(\theta) > 0$, we have

$$g_d(n, \theta; \mathbf{x}) = \frac{1}{\theta^2}(e^{-\theta\mathbf{D}}\mathbf{P}(\Delta))^n e^{-\theta\mathbf{D}}\mathbf{1} - \frac{1}{\theta^2}\mathbf{1} + \frac{\mathbf{x}}{\theta}\frac{1 - e^{(n+1)r\Delta}}{1 - e^{r\Delta}}. \tag{1}$$

(ii) *(Continuously monitored Asian options)*
Let $g_c(t, \theta; \mathbf{x}) := \int_0^\infty e^{-\theta k} v_c(t, k; \mathbf{x}) dk$. Then for any complex θ such that $Re(\theta) > 0$, we have

$$g_c(t, \theta; \mathbf{x}) = \frac{1}{\theta^2} e^{(\mathbf{G}-\theta\mathbf{D})t}\mathbf{1} - \frac{1}{\theta^2}\mathbf{1} + \frac{\mathbf{x}}{r\theta}(e^{rt} - 1). \tag{2}$$

Proof of Proposition 1.

(i) Let $L_d(z, \theta; \mathbf{x}) := \sum_{n=0}^\infty z^n \int_0^\infty e^{-\theta k} v_d(n, k; \mathbf{x}) dk = \sum_{n=0}^\infty z^n g_d(n, \theta; \mathbf{x})$. From Proposition 1(i) and Theorem 2(i) of Cai et al. (2015), for any fixed z and θ such that $|z| < \min\{1, e^{-r\Delta}\}$ and $Re(\theta) > 0$, we have

$$L_d(z, \theta; \mathbf{x}) = \frac{1}{\theta^2}(e^{\theta\mathbf{D}} - z\mathbf{P}(\Delta))^{-1}\mathbf{1} - \frac{1}{\theta^2(1-z)}\mathbf{1} + \frac{\mathbf{x}}{\theta(1-z)(1-ze^{r\Delta})}. \tag{3}$$

By definition, we observe that $g_d(n, \theta; \mathbf{x})$ can be treated as the coefficient of z^n in the power series expansion of $L_d(z, \theta; \mathbf{x})$ with respect to the transform variable z. This motivates us to expand the right-hand side of (3) into a power series of z.

The proof of Theorem 2(i) in Cai et al. (2015) has shown that $e^{\theta\mathbf{D}} - z\mathbf{P}(\Delta)$ is invertible. Moreover, from Corollary 5.6.16 of Horn and Johnson (1985), if there is a matrix norm $||\cdot||$ (without loss of generality, we can take the maximum norm, i.e., $||\mathbf{A}|| = \max\{|a_{ij}|\}$) such that $||\mathbf{A}|| < 1$, then we have $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^\infty \mathbf{A}^k$. In the following, we further narrow the range of z such that $|z| < \min\{1, e^{-r\Delta}, 1/||(e^{\theta\mathbf{D}})^{-1}\mathbf{P}(\Delta)||\}$ and thus the power series expansions with respect to z are well-defined. Then we obtain

$$\begin{aligned}(e^{\theta\mathbf{D}} - z\mathbf{P}(\Delta))^{-1}\mathbf{1} &= (\mathbf{I} - z(e^{\theta\mathbf{D}})^{-1}\mathbf{P}(\Delta))^{-1}(e^{\theta\mathbf{D}})^{-1}\mathbf{1} \\ &= (\mathbf{I} + z(e^{\theta\mathbf{D}})^{-1}\mathbf{P}(\Delta) + \cdots + z^n((e^{\theta\mathbf{D}})^{-1} \\ &\quad \times \mathbf{P}(\Delta))^n + \cdots)(e^{\theta\mathbf{D}})^{-1}\mathbf{1}.\end{aligned} \tag{4}$$

The coefficient of z^n in (4) is $((e^{\theta\mathbf{D}})^{-1}\mathbf{P}(\Delta))^n(e^{\theta\mathbf{D}})^{-1}\mathbf{1}$. Expand the remaining parts in (3) as

$$-\frac{1}{\theta^2(1-z)}\mathbf{1} = -\frac{1}{\theta^2}\mathbf{1}\sum_{i=0}^\infty z^i, \tag{5}$$

and

$$\frac{\mathbf{x}}{\theta(1-z)(1-ze^{r\Delta})} = \frac{\mathbf{x}}{\theta}\sum_{i=0}^\infty z^i \times \sum_{j=0}^\infty z^j e^{jr\Delta}. \tag{6}$$

It is clear that the coefficient of z^n in (5) is $-\frac{1}{\theta^2}\mathbf{1}$, and the coefficient of z^n in (6) is $\frac{\mathbf{x}}{\theta}\frac{1-e^{(n+1)r\Delta}}{1-e^{r\Delta}}$.

By matching the coefficient of z^n in the expansion of $L_d(z, \theta; x)$ to the coefficient of z^n in the definition of $L_d(z, \theta; \mathbf{x}) := \sum_{n=0}^\infty z^n g_d(n, \theta; \mathbf{x})$, we have

$$g_d(n, \theta; \mathbf{x}) = \frac{1}{\theta^2}((e^{\theta\mathbf{D}})^{-1}\mathbf{P}(\Delta))^n(e^{\theta\mathbf{D}})^{-1}\mathbf{1} - \frac{1}{\theta^2}\mathbf{1} + \frac{\mathbf{x}}{\theta}\frac{1-e^{(n+1)r\Delta}}{1-e^{r\Delta}}. \tag{7}$$

From (7) and $(e^{\theta\mathbf{D}})^{-1} = e^{-\theta\mathbf{D}}$ (recall $\mathbf{D}$ is a diagonal matrix), we have proved (1).

(ii) Let $L_c(\mu, \theta; \mathbf{x}) := \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\theta k} v_c(t, k; \mathbf{x}) dk dt = \int_0^\infty e^{-\mu t} g_c(t, \theta; \mathbf{x}) dt$. From Proposition 1(ii) and Theorem 2(ii) of Cai et al. (2015), for any fixed z and θ such that $|z| > r$ and $Re(\theta) > 0$, we have

$$L_c(\mu, \theta; \mathbf{x}) = \frac{1}{\theta^2} m(\mu, \theta; \mathbf{x}) - \frac{1}{\theta^2\mu}\mathbf{1} + \frac{\mathbf{x}}{\theta\mu(\mu - r)}, \tag{8}$$

where $m(\mu, \theta; \mathbf{x}) := (\theta\mathbf{D} + \mu\mathbf{I} - \mathbf{G})^{-1}\mathbf{1}$.

The proof of Theorem 2(ii) in Cai et al. (2015) has shown that $\theta\mathbf{D} + \mu\mathbf{I} - \mathbf{G}$ is invertible. In the following, we focus on such μ with $Re(\mu) > \max\{||\mathbf{G} - \theta\mathbf{D}||, r\}$ to support the definition of an inverse Laplace transform with respect to μ. This further implies $|\mu| > \max\{||\mathbf{G} - \theta\mathbf{D}||, r\}$ and hence the following power series expansion is also well defined. Similar to (4), weobtain

$$\begin{aligned}(\theta\mathbf{D} + \mu\mathbf{I} - \mathbf{G})^{-1}\mathbf{1} &= \frac{1}{\mu}\left(\mathbf{I} - \frac{\mathbf{G} - \theta\mathbf{D}}{\mu}\right)^{-1}\mathbf{1} \\ &= \frac{\mathbf{1}}{\mu} + \frac{(\mathbf{G} - \theta\mathbf{D})\mathbf{1}}{\mu^2} + \cdots + \frac{(\mathbf{G} - \theta\mathbf{D})^n\mathbf{1}}{\mu^{n+1}} + \cdots.\end{aligned} \tag{9}$$

From a property of the Gamma function, we have

$$\int_0^\infty e^{-\mu t}\frac{(\mathbf{G} - \theta\mathbf{D})^i\mathbf{1}}{i!} t^i dt = \frac{(\mathbf{G} - \theta\mathbf{D})^i\mathbf{1}}{\mu^{i+1}}, \quad i = 0, 1, \dots. \tag{10}$$