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Improved integrality gap upper bounds for traveling salesperson problems with distances one and two

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ABSTRACT

We prove new integrality gap upper bounds for the traveling salesperson problem with distances one and two ((1,2)-TSP). We obtain these bounds by investigating the structure of the support graph of optimal solutions to the subtour elimination linear programming relaxation, with one additional cutting plane inequality.

For undirected (1,2)-TSP, our main results are as follows:

- All instances have an integrality gap of at most 5/4.
- Instances admitting half-integral solutions have integrality gap at most 7/6.
- Instances admitting subcubic solutions of cost at most the order of the instance have integrality gap at most 10/9, even without the cutting plane. This bound is tight, and holds in particular for basic solutions in the fractional 2-matching polytope with cost at most the order of the instance.

For directed (1,2)-TSP instances we show an integrality gap upper bound of 3/2 for general instances, and of 5/4 for instances admitting half-integral solutions.

We prove these bounds by providing local search algorithms that, in polynomial time, find 2-matchings with few components in the support of the solution. We show that the run times of our algorithms cannot be considerably improved under standard complexity-theoretic assumptions: we show that finding improved TSP solutions via local search is intractable for edge change parameterized by the size of the neighborhoods even for instances with distances one and two; this strengthens a result of Dániel Marx.

1. Introduction

The traveling salesperson problem (TSP) in metric graphs is one of the most fundamental NP-hard optimization problems. Given an undirected or directed graph G with a metric on its edges, we seek a tour $\mathcal{T}$ (a Hamiltonian cycle) of minimum cost in G, where the cost of $\mathcal{T}$ is the sum of costs of edges traversed by $\mathcal{T}$.

Despite a vast body of research, the best approximation algorithm for metric TSP is still Christofides's (1976) algorithm, which has a performance guarantee of 3/2. Recall that the performance guarantee or approximation ratio of an algorithm for a problem is defined as a number α such that, in polynomial time, the algorithm computes a solution whose value is within a factor α of the optimal value. Generally the bound 3/2 is not believed to be tight. However, the currently largest known lower bound on the performance guarantee obtainable in polynomial time is as low as 123/122 (Karpinski, Lampis, & Schmied, 2013).

One of the most promising techniques to obtain an improved performance guarantee is to use a linear programming (LP) formulation of TSP. Upper bounds on the integrality gap of the LP usually translate to approximation guarantees. In this context, the *subtour elimination relaxation* (SER), or Held–Karp relaxation (Held & Karp, 1970), is particularly important. Its integrality gap is between 4/3 and 3/2, and the value 4/3 is conjectured to be tight (Goemans, 1995). For relevant special cases, the conjecture is known to be true (Boyd, Sitters, van der Ster, & Stougie, 2011; Mömke & Svensson, 2011). It is also known that SER has a close relation to 2-matchings, as was pointed out for instance by Schalekamp, Williamson, and van Zuylen (2014) in the context of perfect 2-matchings.

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1.1. Our contributions

We investigate the structure of the *support graph* of solutions to SER, that is, the graph of edges with non-zero value in an optimal solution to SER. We show how to find a 2-matching (i. e., a collection of paths and cycles) that approximates the minimum number of components in the SER support graph. While we think that our structural findings are of independent interest, they have a direct impact on the integrality gap of SER for TSP with restricted edge weights. In particular, we obtain improved integrality gap upper bounds for the asymmetric and symmetric TSP with distances one and two, a classical and well-studied variant of TSP (Berman & Karpinski, 2006; Bläser & Manthey, 2005; Bläser & Ram, 2005; Karpinski & Schmied, 2012; Papadimitriou & Yannakakis, 1993; Qian, Schalekamp, Williamson, & van Zuylen, 2015; Vishwanathan, 1992; Williamson, 1990). We refer to the symmetric variant (in undirected graphs) as (1, 2)-STSP and to the asymmetric variant as (1, 2)-ATSP .

First, we augment SER by a single cutting hyperplane to a linear program SER$^+$; this way, we enforce that an optimal solution to SER$^+$ takes an integer value. The modification requires an integer cost function, but is not specific to edge costs one and two. We then consider the support graph of an optimal SER$^+$ solution. We show how to modify this graph in such a way that it allows us to prove integrality gap upper bounds for TSP by computing a 2-matching with few components. To this end, we define certain types of improvements that extend the improvements used by Berman and Karpinski (2006) in their approximation algorithm for (1, 2)-STSP. For instance, we show how to transform a 2-matching in the support to another 2-matching not containing isolated vertices without increasing the number of components, in Section 3. With further improvements obtained by applying alternating paths, in Section 5 we find in polynomial time a 2-matching with at most $n/4$ components. This 2-matching then implies an integrality gap upper bound of 5/4 for arbitrary (1, 2)-STSP instances.

Second, we consider half-integral instances of (1, 2)-STSP. A conjecture of Schalekamp et al. (2014) implies that instances exist for which the integrality gap of SER and SER$^+$ is tight and which at the same time are basic solutions to the fractional perfect 2-matching polytope. These basic solutions are well understood and they have a quite specific structure (Balinski, 1965). In particular, they are half-integral (all LP values are multiples of 1/2) and they are subcubic (all degrees in the support are at most three). We show that if the half-integrality part of the conjecture is true, then integrality gap of SER$^+$ is at most 7/6.

Third, we consider the conjecture of Schalekamp et al. (2014) without requiring half-integrality, in Section 4. For (1, 2)-STSP on instances G that admit optimal basic solutions with subcubic support of SER(G) and fractional objective function value $\mathrm{Opt}_{\mathrm{SER}}(G) = |V(G)|$, we obtain a *tight* integrality gap and an improved approximation guarantee of 10/9. We think that the restriction to instances where $\mathrm{Opt}_{\mathrm{SER}}(G) = |V(G)|$ is quite benign since all instances satisfy $\mathrm{Opt}_{\mathrm{SER}}(G) \geq |V(G)|$, and usually the integrality gaps of linear programs decrease with increasing fractional solution costs.

Fourth, we transfer our results from (1, 2)-STSP to (1, 2)-ATSP in a natural way. We prove an SER$^+$ integrality gap upper bound of 3/2 for general instances of (1, 2)-ATSP, and 4/3 if additionally there is a half-integral optimal solution.

Fifth, we show that the SER$^+$ integrality gap upper bound converges to the SER integrality gap upper bound with increasing instance size. By an amplification technique, we use known computational results for small instances to show SER integrality gap upper bounds that differ by less than two percent from our SER$^+$ integrality gap upper bounds. Our results imply a purely computational method that either finds an increased integrality gap lower bound for SER$^+$ or it shows an arbitrarily small difference between the integrality gaps of SER and SER$^+$. Our results for SER provide the currently best integrality gap upper bound for instances with given optimal half-integral solution.

Sixth, we identify structures in (1, 2)-STSP and (1, 2)-ATSP instances that allow us to increase the instance size by an arbitrary factor without decreasing the integrality gap of the instance. One consequence of these results is that there are infinitely many (1, 2)-ATSP instances with integrality gap at least 6/5. The gained insights give raise to conjecture that the integrality gaps of SER and SER$^+$ coincide for both (1, 2)-STSP and (1, 2)-ATSP.

Finally, we strengthen a result of Marx (2008) about finding cheaper solutions to a (1, 2)-STSP instance by local search, in Section 11. Precisely, we show that finding a cheaper solution compared to a given tour by exchanging at most k edges is W[1]-hard, even in undirected TSP instances with distances one and two. This hardness result intuitively says that a brute force search of all subsets of k edges—in time $n^{O(k)}$ for (1, 2)-STSP instances of order n—is essentially optimal, unless many canonical NP-complete problems admit subexponential-time algorithms. Such an intractability result was known before only for TSP instances with *three* distinct city distances, due to Marx (2008). This result suggests that a simple search for local improvements is not efficient. Our proof is similar to that of Marx, with some simplifications.

1.2. Overview of techniques

To show that the subtour elimination support graph contains a 2-matching with few components, we apply a sequence of local improvements. One important step is that we can exclude the solution computed by our algorithm to contain isolated vertices. We use an induction that creates a tree of alternating paths, and show that it is always possible to increase the size of the tree unless there is an improved 2-matching (i. e., with fewer components).

All five of our integrality gap upper bound results use an accounting technique that distributes an amount of n coins to components of the 2-matching. We assign a sufficient amount of coins to each component of the 2-matching to ensure that the total number of components cannot exceed the aimed-for upper bound. However, we employ two entirely different schemes in order to provide a distribution of coins. Our result for degree-3-bounded support graphs initially assigns one coin to each vertex, and then redistributes the coins to the components. A similar accounting technique has been used by Berman and Karpinski (2006). However, we exploit properties of the subtour elimination constraints in order to ensure the existence of fractional coins that are available provided that no local improvements are possible.

The remaining integrality gap upper-bound results use the LP values of the subtour elimination relaxation directly. One way to interpret the technique is to initially distribute n coins to *edges*, where each edge obtains a fraction of the coin according to its LP value. As key idea, we formulate the distribution of values to components via a new linear program. The linear program takes the SER solution x^* and the aimed-for number of coins per component as parameter; this way, we reduce the analysis to finding a feasible solution to the linear program. To find a feasible solution to the new linear program, we split the edges of the SER support graph into sub-edges such that each sub-edge e has the same LP value x^*_e. In the resulting multigraph, we obtain a collection of disjoint alternating paths of which certain types then lead to improved 2-matchings.

1.3. Related work

Both (1, 2)-STSP and (1, 2)-ATSP are well-studied from the approximation point of view. For (1, 2)-STSP, it is NP-