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A comparative study of the arcflow model and the one-cut model for one-dimensional cutting stock problems

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We consider the one-dimensional cutting stock problem which consists in determining the minimum number of given large stock rolls that has to be cut to satisfy the demands of certain smaller item lengths. Besides the standard pattern-based approach of Gilmore and Gomory, containing an exponential number of variables, several pseudo-polynomial formulations were proposed in the last decades. Much research has dealt with arcflow models, their relationship to the standard model, and possible reduction methods, whereas the one-cut approach has not attracted that much scientific interest yet.

In this paper, we aim to compare both alternative formulations from a theoretical and numerical point of view. As a theoretical main contribution, we constructively prove the equivalence of the continuous relaxations of the one-cut model, the arcflow model, and the pattern-based model. In particular, the relationship between the one-cut model and the pattern-based model has remained an open question since the one-cut approach was proposed.

Moreover, in order to make a computational comparison of the arcflow model and the one-cut model, we present how reduction methods, partly originating from arcflow considerations, can successfully be transferred to the one-cut context. Furthermore, we derive relations between the numbers of variables and constraints in both models, and investigate their influences in numerical simulations.

1. Introduction

The term *cutting and packing* addresses a wide variety of (mostly $\mathcal{NP}$-hard) combinatorial optimization problems with high relevance in many fields of practice. Despite containing a large number of very diversified problem formulations, many of them share a common basic structure (Oliveira, 2016): select a subset of (small) items and assign it to larger objects by respecting some (geometrical) constraints. Note that, from a pure mathematical point of view, *cutting* and *packing* are often closely related since they principally describe the same process, but from a different perspective. Nevertheless, both terms are well-established in literature to better refer to the particular application where the considered problem comes from. In most cases, the general objective is to minimize the waste, i.e., the portion of material that cannot be used for the intended purpose. More precisely, based on the typology of Wäscher, Haußner, and Schumann (2007), (almost) all cutting and packing problems can be classified into:

- *Input minimization problems*, where as few resources as possible shall be used to achieve a predefined goal, and
- *Output maximization problems*, where given (limited) resources shall be used to provide the best possible result.

As the case may be, the economic component (i.e., lower production costs in a broad sense) and sustainability issues (i.e., less waste of raw material in general) of all these tasks are obvious. Indeed, some by far not exhaustive applications stressing this relevance can be found in Gradišar, Jesenko, and Resinovič (1997), Kallrath, Rebennack, Kallrath, and Kusche (2014), Koch, König, and Wäscher (2009), Morabito and Garcia (1998) and Stadtler (1990).

In this paper we focus on the one-dimensional cutting stock problem (CSP). The CSP is one of the most important representatives in combinatorial optimization (see Delorme, Iori, and Martello (2016, Fig. 1) for the trend of related publications); the study of its structure and applications already started in 1939, when Kantorovich (1939) formulated the first model to cope with that problem. Therein, based on an upper bound for the number of stock rolls, an assignment model with binary and integer

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variables is proposed. Unfortunately, this approach has turned out to possess two major drawbacks: a very weak continuous relaxation (Martello & Toth, 1990), and a huge number of symmetries arising from permutations. Both points may be very critical when trying to tackle this model by means of branch-and-bound techniques, see Valério de Carvalho (1999) or Vance (1998); hence most authors forbear from doing further research regarding this approach.[1] In 1961, Gilmore and Gomory (1961) introduced a pattern-based approach, hereinafter referred to as the *standard model* (see also Section 2), that overcomes these disadvantages.

A way to tackle the integer problem consists in the consideration of other modeling approaches, most notably the arcflow model (Valério de Carvalho, 2002) and the one-cut model (Dyckhoff, 1981). In the last years, much research has been done to investigate and improve especially arcflow models (Brandão & Pedroso, 2016) for the CSP and algorithms (Brandão, 2016), even with respect to related problems. Contrary to that, the one-cut model (Dyckhoff, 1981) has attracted significantly less scientific interest so far. Indeed, the only important contribution known in the literature is due to Stadtler (1988), who investigated the structure of the original model in more detail, and presented an extension to Dyckhoff's original approach. In that paper the author stated:

> "This leads to the conclusion, that all cutting stock problems that can be solved by one-cut models could also be solved by the column generation approach for the complete-cut model (but not vice versa). [...] The set of real world cutting stock problems solvable by the one-cut model is only a subset of those which could be tackled by the column generation approach."

Moreover, Valério de Carvalho (2002) pointed out:

> "The number of variables in one-cut models is pseudo-polynomial, and does not grow explosively as in the classical approach. That does not mean that the model is amenable to an exact solution by a good integer LP code, due to the symmetry of the solution space. [...] To our knowledge, the integer solution of one-cut models [...] has never been tried."

At that time, computers were much slower than they are today, and addressing the exact solution of CSP was not an easy task. Valério de Carvalho (1999) was able to solve exactly large instances, including some open instances of the bin-packing problem, because he implemented a column (and row) generation algorithm that reduced the computational burden substantially. Computational times to enumerate all the variables and constraints and to solve then the complete models exactly were unreasonably large. This fact and the statements above, despite being probably directed to more general cutting stock problems, may have contributed to almost excluding the one-cut model from further scientific discussions regarding the cutting stock problem. Nevertheless, the main reason that discouraged researchers was possibly the fact that the papers (Dyckhoff, 1981; Stadtler, 1988), published in 1981 and 1988, respectively, had no (theoretical) results concerning the quality of the bound provided by the continuous relaxation of the one-cut model. Mainly after the publication of the book by Nemhauser and Wolsey (1988), in 1988, the strength of the models became a central concern in IP modeling.

In this paper, we provide a constructive proof for the equivalence of the (reduced) one-cut model, the (reduced) arcflow model, and the classical pattern-based approach that also holds for the respective continuous relaxations. Observe that this relationship (for the LP models) has not been formally established before in the literature. It shows that the one-cut model is a strong model, which is worth exploring. Therefore, in a first step, we propose several reduction methods that are useful to decrease the numbers of variables and constraints, as well as the symmetry of one-cut models, which are also a contribution of this paper. Based on these reductions, the (reduced) one-cut model is shown to differ from the (reduced) arcflow model by exactly m fewer variables and constraints.

Our numerical simulations show that, contrary to Stadtler's (1988) statement, nowadays, we can solve a representative set of (reduced) full one-cut models with the current state-of-the-art ILP software packages and computers in reasonable time. When compared to the arcflow model, as both models are equally strong and similar in size, the computational times are comparable. Furthermore, as the one-cut model has fewer variables and constraints, its performance is slightly better than that of the arcflow model. It is not the purpose of this paper to present a competitive and thorough computational study involving all available codes and approaches from the literature, but to put the one-cut model in its true light when lastly comparing its complexity to that of other well-known formulations for the cutting stock problem.

This paper is organized as follows: in the next section, we state the general assumptions and review the standard Gilmore and Gomory model. Then, Section 3 starts with a repetition of the arcflow model (Valério de Carvalho, 2002), and proposes some important reduction methods that can be applied in that context. Further (more complex) powerful reduction methods for arcflow models and possible generalization are discussed in Brandão and Pedroso (2016). Afterwards, we show how the introduced techniques can successfully be applied to the one-cut approach. In Section 4, we provide several theoretical results to compare both improved formulations: on the one hand with respect to the performance of the LP relaxation, and on the other hand with respect to the numbers of variables and constraints. In a final step, we investigate the effects of these differences by means of numerical simulations and give some conclusions.

2. General assumptions and the standard model

The one-dimensional cutting stock problem aims at finding the minimum number of large stock rolls of length L, that has to be cut to satisfy the demands b_i of given smaller item lengths l_i ($i \in I := \{1, \dots, m\}$). By introducing $l = (l_1, \dots, l_m)^\top$ and $b = (b_1, \dots, b_m)^\top$ we can refer to a specific CSP by means of the *instance* $E = (m, l, L, b)$. Without loss of generality, we may assume:

(i) All input data are positive integers.
(ii) The item lengths are of strictly decreasing order, i.e., $L > l_1 > l_2 > \dots > l_m > 0$. If such an order is not given from the beginning, we can easily obtain it by a sorting algorithm, e.g. merge sort, in $\mathcal{O}(m \cdot \log m)$ operations.
(iii) The equation $L = \max \left\{ l^\top a \mid a \in \mathbb{Z}_+^m,\ l^\top a \le L,\ a \le b \right\}$ is satisfied.[2] Otherwise L could be shortened.
(iv) The inequality $l_1 + l_m \le L$ holds. Otherwise, it is possible to discard the items of length l_1, if the optimal objective value of the remaining instance is increased by b_1.

Thereby, the set of *(cutting) patterns* for a given instance $E = (m, l, L, b)$ results to $P(E) = \left\{ a \in \mathbb{Z}_+^m \mid l^\top a \le L \right\}$. More precisely, only *maximal cutting patterns*, described by

$$P^\star := P^\star(E) := \left\{ a \in P(E) \mid l^\top a + l_m > L \right\},$$

[1] Nevertheless, the scientific work of Kantorovich in the field of optimally allocating given resources has been honored with the Nobel prize in economics (1975).

[2] Note that $a \le b$ means that $a_i \le b_i$ holds for all $i \in I$.