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On the price of anarchy in a single-server queue with heterogeneous service valuations induced by travel costs[☆]

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ABSTRACT

This work presents a strategic observable model where customer heterogeneity is induced by the customers' locations and travel costs. The arrival of customers with distances less than x is assumed to be Poisson with rate equal to the integral from 0 to x, of a nonnegative intensity function h. In a loss system M/G/1/1 we define the threshold Nash equilibrium strategy x_e and the socially-optimal threshold strategy x^*. We investigate the dependence of the price of anarchy (PoA) on the parameter x_e and the intensity function. For example, if the potential arrival rate is bounded then PoA is bounded and converges to 1 when x_e goes to infinity. On the other hand, if the potential arrival rate is unbounded, we prove that x^*/x_e always goes to 0, when x_e goes to infinity and yet, in some cases PoA is bounded and even converges to 1; if h converges to a positive constant then PoA converges to 2; if h increases then the limit of PoA is at least 2, whereas if h decreases then PoA is bounded and the limit of PoA is at most 2. In a system with a queue we prove that PoA may be unbounded already in the simplest case of uniform arrival.

1. Introduction

Customers of a service system often have heterogeneous service valuations, and this heterogeneity may be caused by various reasons. In this paper we study a model with customers located at different distances from the service facility, and therefore incur different "travel costs". Of course "location" may refer to a geographic location or it may serve as a metaphoric way expressing different preferences on the ideal type of service. We study the price of anarchy (PoA) comparing the social welfare produced by the system under socially-optimal and individually-optimal customer behavior. In our model, the potential arrival rate of customers at distance at most x is defined as the integral of a given density function. This structure of heterogeneity enables us to obtain interesting conclusions on the PoA.

The performance of service systems with strategic customers has attracted much attention in recent years (see, for example, Hassin, 2016; Hassin & Haviv, 2003). Naor (1969) was the first to introduce a queueing model that describes customer rational decisions. The model considers an FCFS M/M/1 system with homogeneous customers, a fixed reward associated with service completion, and linear waiting costs. The Nash equilibrium solution in this model is simple since there exists a dominant pure threshold strategy n_e, such that an arriving customer joins the queue if and only if the observed queue upon arrival is shorter than n_e. This strategy maximizes the individual's expected welfare regardless of the strategies adopted by the others. The socially-optimal behavior is characterized by a pure threshold strategy n^* such that $n^* \le n_e$.

The *price of anarchy* measures the inefficiency of selfish behavior. It is defined as the ratio of the social welfare under optimum to the social welfare in equilibrium.

Naor assumes that customers are homogeneous with respect to service valuation, and much of the literature on observable queues (i.e., assuming customers know the queue length before joining it) follow this assumption. Some exceptions are described in Section 2.5 of Hassin and Haviv (2003). For example, Larsen (1998) assumes that the service value is a continuous random variable and proves that the profits and social welfare are unimodal functions of the price. For the case of a loss system (where customers join only if the server is idle) Larsen proves that the profit-maximizing fee exceeds the socially optimal fee. Miller and Buckman (1987) consider an M/M/s/s loss system with heterogeneous service values and characterize the socially optimal fee.

Some authors recently investigated the PoA in various service systems (see, for example, Hassin & Kleiner, 2011; Wang, Zhang, & Huang, 2017 and Section 5.8 in Hassin, 2016). Most relevant to

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our work is Gilboa-Freedman, Hassin, and Kerner (2014) where the PoA in Naor's model is shown to have an odd behavior. It increases sharply (from 1.5 to 2) as the arrival rate comes close to the service rate and becomes unbounded exactly when the arrival rate is greater than the service rate, which is odd since the system is always stable.

In this paper we introduce heterogeneity in service valuation through a Hotelling-type model where customers reside in a "linear city" and incur "transportation costs" from their locations to the location of the server. Similar models have been investigated (e.g. D'aspremont, Gabszewicz, & Thisse, 1979; Dobson & Stavrulaki, 2007; Economides, 1986; Gallay, Olivier, & Hongler, 2008; Hotelling, 1929; Kwasnica & Euthemia, 2008; Pangburn & Stavrulaki, 2008; Ray & Jewkes, 2004 and Sections 6.7 and 7.5 in Hassin, 2016) but they all assume a constant density (possibly restricted to an interval). In contrast, we allow non-uniform distributions of customer locations, and the potential arrival of customers with distances less than x from the service facility is with rate $\lambda(x) = \int_0^x h(y)dy < \infty$, where $h(y)$ is a nonnegative "intensity" function of the distance y. The intensity function and (linear) travel costs jointly generate the distribution of customer service valuations. A simple example is a two-dimensional city, in which the arrival of customers is uniform. In this case the intensity function can be defined as $h(x) = 2\pi x$, and so the arrival of customers with distances less than x is assumed to be a Poisson process with rate $\lambda(x) = \int_0^x 2\pi y dy = \pi x^2$.

We first consider an M/G/1/1 loss system and define x_e as the threshold Nash equilibrium strategy, namely the maximal distance from which customers will enter service under individual optimality, and x^* as the threshold value that attains optimal social welfare. We show how x_e is determined by the parameters R, μ, c_t and c_w, and the function h, of the model.

We prove that PoA $\to 1$ when $x_e \to 0$. The behavior of PoA when $x_e \to \infty$ is more complex and interesting:

- The PoA limit does not always exist, and it may be infinite.
- If the potential amount of customers arriving from long distances is small (i.e., $\int_0^\infty h(y)dy < \infty$), then in the limit there is no difference between the social and equilibrium optimal benefits, namely $\lim_{x_e\to\infty} \text{PoA}(h, x_e) = 1$, (even though the corresponding optimal strategies x^* and x_e do not coincide).

The rest of the paper is dedicated to the case in which $\int_0^\infty h(y)dy = \infty$.

- We develop an explicit formula to calculate $\lim_{x_e\to\infty} \text{PoA}(h, x_e)$ when it exists and show that if h, h' are monotonic (where h' is the derivative of h), then this limit exists and we arrive at a very simple formula to calculate it.
- If h converges to a constant then $\lim_{x_e\to\infty} \text{PoA}(h, x_e) = 2$.
- If h decreases (increases) monotonically and $\lim_{x_e\to\infty} \text{PoA}(h, x_e)$ exists, then $\lim_{x_e\to\infty} \text{PoA}(h, x_e) \le 2$ (≥ 2).
- For any two nonnegative intensity functions h_1, h_2 s.t. $h_1/h_2 \to c > 0$, if the corresponding $\lim_{x_e\to\infty} \text{PoA}(h_i, x_e)$, $i = 1, 2$, exist, then $\lim_{x_e\to\infty} \text{PoA}(h_1, x_e) = \lim_{x_e\to\infty} \text{PoA}(h_2, x_e)$.
- If h_1, h_2, h_1', h_2', are all monotonic and from some point on $h_1 \le h_2$, then the corresponding $\lim_{x_e\to\infty} \text{PoA}(h_i, x_e)$, $i = 1, 2$, exist, and $\lim_{x_e\to\infty} \text{PoA}(h_1, x_e) \le \lim_{x_e\to\infty} \text{PoA}(h_2, x_e)$.

All the above results relate to a loss system. In the last section of this work, we turn our attention to a system with a queue. We prove that in this model, the price of anarchy may be unbounded already in the simple case of constant intensity.

2. Model description

Consider a single-server queue located at the origin. We make the following assumptions:

1. For all $x \ge 0$, customers with distances less than x, arrive to the system according to a Poisson process with rate $\lambda(x) = \int_0^x h(y)dy$, where $h(y)$ is a nonnegative "intensity" function defined for all $y \ge 0$, s.t. $0 < \lambda(x) < \infty$ for all $0 < x < \infty$.
2. Customers know their distance from the server.
3. The queue length is observable.
4. Customers are risk neutral, maximizing expected net benefit.
5. The service distribution is general with average rate μ.
6. The benefit from a service is R.
7. There is a waiting cost c_w per unit time while in the system.
8. There is a traveling cost of c_t per unit distance. Traveling is instantaneous. (If $c_t = 0$, we obtain Naor's model with rate $\lambda = \int_0^\infty h(y)dy$.)
9. $\nu = \frac{R\mu}{c_w} > 1$.
10. The decision of the customer is whether to join the queue or balk.

3. A loss system

First, we consider an M/G/1/1 loss system. Namely, it is not possible to wait for service. The optimal strategy of a customer located at a distance x from the origin, is to enter service if the server is idle and $R \ge \frac{c_w}{\mu} + c_t x$. Consequently, the threshold strategy

$$x_e = \frac{R - c_w/\mu}{c_t} \tag{1}$$

is the unique Nash equilibrium strategy. Under this strategy, a customer located at a distance x, enters service iff the server is idle and $x \le x_e$.

The utility of a customer entering service from location x is:

$$R - c_w/\mu - c_t x = c_t(x_e - x). \tag{2}$$

Define $\rho(x)$, the average server utilization when the arrival threshold is x, as

$$\rho(x) = \frac{\lambda(x)}{\mu} = \frac{1}{\mu}\int_0^x h(y)dy.$$

The probability $\pi_0(x)$, of an idle server satisfies:

$$\pi_0(x)\lambda(x) = (1 - \pi_0(x))\mu.$$

This implies that:

$$\pi_0(x) = \frac{1}{1+\rho(x)} = \frac{1}{1 + \frac{1}{\mu}\int_0^x h(y)dy}.$$

By (2), the expected social benefit per unit of time associated with threshold x satisfies

$$S(x) = c_t\int_0^x (x_e - y)h(y)\pi_0(x)dy = \frac{c_t\int_0^x (x_e - y)h(y)dy}{1 + \frac{1}{\mu}\int_0^x h(y)dy}. \tag{3}$$

Let x^* be the threshold value that maximizes social welfare. Note that x_e can be viewed as a parameter that is determined by the primitive parameters of the model such that $x_e = \frac{R\mu - c_w}{c_t\mu}$ (see (1)), whereas x^* is implicitly defined as a solution of an equation (see Eq. (7) below) involving x_e. Hence we relate to x^* as a function of x_e.

Proposition 3.1. *For every $x_e \ge 0$ the optimal threshold strategy x^* is unique and satisfies,*

$$x^* \text{is a continuous strictly increasing function of } x_e. \tag{4}$$

$$\lim_{x_e\to\infty} x^* = \infty. \tag{5}$$