European Journal of Operational Research 254 (2016) 656–666

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Interfaces with Other Disciplines

# An improved method for pricing and hedging long dated American options

Frank J. Fabozzi[a,*], Tommaso Paletta[b], Silvia Stanescu[b], Radu Tunaru[b]

[a] EDHEC Business School, Nice cedex 3 BP3116 06202, France
[b] Business School, University of Kent, Canterbury CT2 7PE, UK

ARTICLE INFO

*Article history:*
Received 2 February 2015
Accepted 2 April 2016
Available online 9 April 2016

*Keywords:*
American options
Optimal exercise price
Quasi-analytic method
Delta-hedging
LEAPS

ABSTRACT

The majority of quasi-analytic pricing methods for American options are efficient near maturity but are prone to larger errors when time-to-maturity increases. We introduce a new methodology to increase the accuracy of almost any existing quasi-analytic approach in pricing long-maturity American options. The new methodology, called the "extension-method", relies on an approximation of the optimal exercise price near the beginning of the contract combined with existing pricing approaches so that the maturity range for which small errors are attainable is extended. Our method retains the quasi-analytic nature of the methods it improves. Generic quasi-analytic formulae for the price of an American put as well as for its hedging parameter are derived. Our scenarios-based numerical study indicates that our method considerably improves both the pricing and the hedging performance of a number of established approaches for a wide range of maturities. The superiority of this approach is illustrated with real financial data by considering S&P 100™ LEAPS® options traded from January 2008 to May 2015.



## 1. Introduction

The problem of pricing American options has been widely examined in the last 40 years. The main challenge is due to the fact that the American optionality requires the selection of the optimal exercise price (henceforth, OEP) together with the valuation of the contingent claim. Several types of approximation approaches have been proposed in the literature to solve this problem. Within the broad class of approximation methods, in this paper we focus on the quasi-analytic methods consisting of analytic formulae that require at most a reasonably small number of numerical solutions of integral equations. The first method in this subclass is described in Geske and Johnson (1984), who used a portfolio of compound European options to replicate the early exercise feature of American options. Bunch and Johnson (1992) improved the efficiency of the Geske–Johnson method by optimally locating the exercise points and showed that most of the time only two – and in a few cases for deep-in-the-money options only three – early-exercise dates including maturity are required. The quadratic approximation in Barone-Adesi and Whaley (1987) gives an approximated solution of the Black–Scholes partial differential equation in closed form. This method, extremely fast and accurate for very short and very long maturities, has been refined by Ju and Zhong (1999) including a second-order extension that improves accuracy for middle-term maturities. Subsequently, Li (2010a) further refined the Ju-Zhong method by a more careful use of the smooth pasting condition for American options. Li's method results in a more precise estimation of the OEP. However, the approximations in Barone-Adesi and Whaley (1987); Ju and Zhong (1999) and Li (2010a) have the limitation that the pricing error cannot be controlled, that is, these methods are not convergent to the "true" price because they cannot be made more precise by including additional terms. Ju (1998) proposed a piecewise exponential function for the OEP.

An important step in the American option pricing literature was the result of Kim (1990) and Carr, Jarrow, and Myneni (1992), who derived an implicit-form integral equation for the OEP. Hence, the pricing of American options can be reduced to identifying the OEP efficiently. Ibáñez (2003) modified Kim's method to guarantee that the prices monotonically converge to the true prices when the number of time steps increases. Kim, Jang, and Kim (2013), based on an idea from Little, Pant, and Hou (2000), transformed the integral equation into a numerical functional form with respect to the optimal exercise boundary, and subsequently constructed an iterative method to calculate the boundary as a fixed point of the functional. Recently, Broadie and Detemple (1996), Laprise, Fu, Marcus, Lim, and Zhang (2006) and Chung, Hung, and Wang (2010) proposed tight quasi-analytic bounds for American options.

* Corresponding author. Tel.: +1 215 598 8924.
*E-mail address:* frank.fabozzi@edhec.edu, fabozzi321@aol.com (F.J. Fabozzi).

http://dx.doi.org/10.1016/j.ejor.2016.04.002
0377-2217/

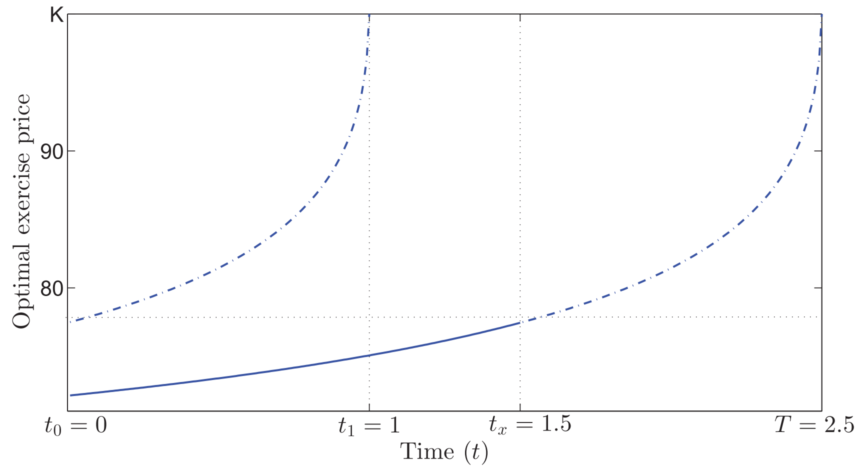


**Fig. 1.** Example of extension method mechanism.
*Note:* The optimal exercise prices of two American put options are considered in the figure. The two options are written on the same underlying asset with $\sigma = 20$ percent, $\delta = 5$ percent, $r = 8$ percent and $K = 100$. One option has maturity $t_1 = 1$ year and the other $T = 2.5$ years. The continuous line represents the optimal exercise price of the option with maturity $T$ and the dash-dot lines represent the optimal exercise of the option with maturity $t_1$. In particular, the left-most dash-dot line is the 'original' function and the other is its translation over the continuous line to show they coincide in the interval $[t_x, T]$ where $t_x = t_0 + (T - t_1) = 1.5$ years represents the size of the translation. The OEPs are calculated by the integral method in Kim (1990).

Additionally, Chockalingam and Muthuraman (2015) employed the approximate moving boundaries method which iteratively finds an approximation of the OEP and Chockalingam and Feng (2015) extended on Ibáñez and Paraskevopoulos (2011) to investigate the cost of a suboptimal OEP.

Almost all the methods may produce large pricing errors for long-maturity options since the convergence to the "true" price depends on the decrease of the size of the time-step (i.e. early exercise dates) or, equivalently, on the increase in the number of iterations. However, an increase in the number of iterations makes these methods rapidly inefficient. In Table 1 – see rows 'Std', which contain the results for the 'standard' versus rows 'Ext', the 'extended' version of the methods – the performance of several pricing methods is reported with respect to the mean absolute percentage error, MAPE, for maturities ranging from less than six months to between four and a half and five years. All the methods considered were selected because they perform very well for short maturities as reported by several other studies.[1]

In this paper, we propose a quasi-analytic approach that aims to improve the performance of existing methods in pricing and hedging long-maturity options. The new approach, which we call the "extension method", relies on the fact that the OEP is independent of the current underlying asset price. The state space is divided into the continuation and the exercise regions, which are precisely separated by the OEP. In the following, in a novel way each option's time-to-maturity is divided into two components according to the closeness to the maturity date. We use a constant approximation function[2] for the first part of the option life and existing pricing methods (with their associated estimation for the OEPs) for the second part (see Fig. 1). The division of time to maturity and OEP profile is marked by a time-point $t_x$. The value of $t_x$ is determined by performing several empirical trials and, although it is dependent on the quasi-method, our results in Fig. 2 suggest that $t_x/T$ is around 0.5, although for Ju and Zhong (1999) $t_x/T = 0.3$ and for Ibáñez (2003) $t_x/T = 0.35$.

Under the proposed extension methodology, the option price is equal to the sum of the expected discounted-payoff from the first part of the option life and the expected discounted-payoff from the second part, conditioned on not exercising the option in the first part. We derive formulae for the American put price and also for the corresponding hedging parameters.[3] We also prove the convergence of the American put option price obtained with our proposed extension method to the perpetual put price when maturity increases infinitely. An extensive scenario-based study is carried-out showing that, when compared with established quasi-analytic methods, the new method leads to sizeable improvements in pricing and hedging American options, especially for longer maturities where existing methods generally fail. Then, we show that the extension method also improves the existing methodologies when applied to real data, LEAPS® options on the S&P 100™ index between 2 January 2008 and 29 May 2015.

The remainder of the paper is organized as follows. Section 2 describes the modelling framework. The main theoretical results are discussed in Section 3 where the closed-form pricing and hedging formulae are derived. Section 4 is a numerical scenarios-based study of the pricing and hedging performance of the extension method. Section 5 reports the pricing performance over the S&P 100™ LEAPS® options and Section 6 concludes.

## 2. Modelling framework

All modelling referring to American option pricing in this paper is done assuming that, under the risk-neutral measure $\mathbb{Q}$, the dynamics of the underlying stock $S$ is given by:

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t, t \geq t_0 \qquad (1)$$

where $r$ is the risk-free rate and $\delta$ is the annual dividend yield with continuous compounding. For simplicity, the difference $r - \delta$ is denoted henceforth by $b$ and $\{W_t\}_{t \geq t_0}$ refers to a Wiener process under the martingale measure $\mathbb{Q}$.

Without any loss of generality, we only consider the case of American put options.[4] The OEP of the American put option with maturity $T$ and strike price $K$ is a continuous function, see Jacka (1991), non-decreasing with respect to time, with limiting value for

[1] The comparison is done by using the studies reported in Table 1.

[2] The fact that the OEP becomes constant for long maturities helps with the iterative methods as well; since the boundary at time $t_n$ is a good starting point for the boundary at $t_{n-1}$. We thank an anonymous referee for making this point.

[3] The formulae are given in the on-line appendix.

[4] All the formulae and propositions can be equivalently derived for American call options. Additionally, one can price and hedge call options by using the put-call symmetry in McDonald and Schroder (1998).