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Offsetting inventory replenishment cycles

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A B S T R A C T

The inventory-staggering problem is a multi-item inventory problem in which replenishment cycles are scheduled or offset in order to minimize the maximum inventory level over a given planning horizon. We incorporate symmetry-breaking constraints in a mixed-integer programming model to determine optimal and near-optimal solutions. Local-search heuristics and evolutionary polishing heuristics are also presented to achieve effective and efficient solutions. We examine extensions of the problem that include a continuous-time framework as well as the effect of stochastic demand.

1. Introduction

Much of the inventory-research literature assumes the replenishment and stocking of an item is independent of that for other items. This may be a valid assumption if there are no restrictions on storage space, capital requirements, transportation capacities, or any other required resources. In the presence of such constraints, however, determining the inventory policies of each item independently may call for replenishment quantities that ultimately violate these restrictions. Thus, the coordinated replenishment of multiple items becomes a practical necessity.

A traditional textbook approach to the deterministic, multi-item problem involves the use of Lagrange multipliers within the economic order quantity (EOQ) calculation (see, for example, Hadley & Whitin, 1963). This effectively increases the holding costs to reduce the resulting order quantities, thus satisfying the resource restriction (Nahmias & Olsen, 2015). However, a shortcoming of the Lagrangian approach is that it effectively assumes all items will be received simultaneously at some point in the planning horizon; consequentially, the maximum resource requirement will occur at that time.

Another prevalent approach to the constrained, multi-item problem is to stagger, or offset, the orders in order to avoid the simultaneous receipt of orders as much as possible (see De Schrijver, Aghezzaf, & Vanmaele, 2013 for a recent review of constrained multi-item inventory systems). Much of the early research in this area assumes that all items have a common order cycle length (e.g., Homer, 1966, Page & Paul, 1976, Rosenblatt & Rothblum, 1990, Zoller, 1977). Other research has focused on developing bounds for the problem (e.g., Gallego, Queyranne, & Simchi-Levi, 1996, Hum, Sharafali, & Teo,, 2005). Gallego, Shaw, and Simchi-Levi (1992) have shown the offsetting problem to be strongly NP-complete.

Research on the offsetting problem without restricting the order cycle lengths has been considerably less prevalent. Murthy, Benton, and Rubin (2003) considered the basic offsetting problem of minimizing the maximum resource requirements for situations in which the order cycle lengths of the items are known and are allowed to be independent of one another. They provided an optimal offsetting procedure for two items and a heuristic procedure for offsetting more than two items. Subsequent studies have focused on the development of heuristics to solve this problem, including genetic algorithms (Moon, Cha, & Kim, 2008, Yao & Chu, 2008), a smoothing procedure utilizing a Boltzmann function (Yao, Chu, & Lin, 2008), and local-search procedures (Croot & Huang, 2013). Boctor (2010) presented a mathematical formulation to solve small instances of the problem—up to 20 items—with commercially-available software and developed a hybrid heuristic and a simulated-annealing algorithm to solve larger instances of the problem.

The purpose of this research is twofold. First, we generalize the existing literature by utilizing symmetry reduction to reduce the problem size in order to optimally solve larger problems than previously achieved and by proposing heuristics that we show to provide better solutions than previous methods. Then, we extend the current research by analyzing continuous-based replenishment systems as well as stochastic demand. In the next section, we present the problem definition and the mixed-integer programming formulation of Boctor (2010) as well as a description of the data used to analyze our results. We then discuss the

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symmetry-reduction approach in Section 3 and provide experimental results quantifying the effect on computational requirements. Two types of heuristics are utilized, and their performance is analyzed in Section 4. Further investigations into the offsetting problem are presented in Section 5, including a continuous-time model and the effect of stochastic demand. Finally, we present the conclusions of our research.

2. Problem description

Consider the situation in which there are a number of items ($i = 1, \ldots\ldots, N$) to be stocked, each with a deterministic demand rate, d_i, and a deterministic order cycle, k_i (and, hence, a known order quantity, $q_i = d_i k_i$). The demand rates of the items are assumed to be independent of one another, as are the order cycle lengths. Lead time is also assumed to be deterministic—as noted by Murthy et al. (2003), we can set it to zero without loss of generality—with instantaneous replenishment. The planning horizon spans T time periods ($t = 0, \ldots\ldots, T-1$). Even with an infinite time horizon (as $T \to \infty$), the length of the planning horizon would need to be no more than the least common multiple of the order cycle lengths (the cycles then repeat); from a practical perspective, though, the planning horizon for which the parameters are known and constant (demand, order cycles, product mix, etc.) may be somewhat less. Finally, the objective is to identify when each item should be replenished in order to minimize the maximum resource requirement (for ease of exposition, we will present the problem in terms of the maximum inventory level for the remainder of the analysis), S, at any time during the planning horizon.

Boctor (2010) presented a mixed-integer program for this problem, in which the decision variables, X_{ij}, are defined to be equal to 1 if item i is first replenished at time j ($j = 0, \ldots\ldots, k_i - 1$). The mixed-integer programming formulation is then:

$$\text{Minimize} \quad S \tag{1}$$

$$\textit{Subject to}: \quad \sum_{j=0}^{k_i-1} X_{ij} = 1 \quad \text{for } i = 1, \ldots, N \tag{2}$$

$$\sum_{i=1}^{N} \sum_{j=0}^{k_i-1} s_{ijt} X_{ij} \le S \quad \text{for } t = 0, \ldots, T-1 \tag{3}$$

$$S \ge 0, \quad X_{ij} \in \{0, 1\} \quad \text{for } i = 1, \ldots, N\ j = 0, \ldots, k_i - 1 \tag{4}$$

where $s_{ijt} = q_i - d_i \tau_{ijt}$ is the inventory level of item i at time t, and $\tau_{ijt} = (k_i + t - j) \bmod (k_i)$ is the time elapsed since the most recent replenishment. This formulation requires $\sum_i k_i$ binary variables, one continuous variable, and $N + T$ constraints. For example, the MIP formulation for the 9-item problem of Murthy et al. (2003) would require 73 binary variables.

To evaluate the effect of symmetry reduction and the quality of the heuristics developed, data sets ranging from 9 to 500 items are used in the following sections. The 9-item example is taken directly from Murthy et al. (2003). Demand rates for 20-, 50-, and 200-item instances were generated using the approach of Boctor (2010) in which demand is uniformly distributed between 5 and 30 units per period. For the larger data sets, Boctor (2010) generated demand rates for each item from one of three ranges: uniformly distributed from 5 to 30 units, from 50 to 100 units, and from 150 to 200 units; we do this for 100-, 200-, and 500-item instances. We generated order cycle lengths randomly selected from the divisors of 360 between 2 and 20—to maintain a planning horizon of 360—for all sizes of problems; to evaluate the heuristics, we also follow the approach of Boctor (2010) for 200-item instances in which the order cycle length is uniformly distributed between 2 and 12 (as this provides a planning horizon of 27,720, it is not used in our optimization experimentation). Table 1 provides a summary of the data used in the subsequent analyses.

Table 1
Summary of data used in analyses.

Number of items	Demand rate		Order cycle length	
	U(5, 30),	U(5, 30) U(50, 100), U(150, 200)	Divisors of 360	U(2, 12)
9†				
20	√		√	
50	√		√	
100		√	√	
200*a*	√		√	
200*b*		√	√	
200*c*	√			√
200*d*		√		√
500		√	√	

Note: †Data taken from Murthy et al. (2003).

3. Optimization through symmetry reduction

As is typical with many binary programs, the solution time required to identify an optimal solution increases rapidly with respect to the problem size. The mixed-integer programming formulation presented above requires $K = \sum_i k_i$ binary variables; thus, there will be 2^K solutions, of which $\prod_i k_i$ are feasible. Lower bounds can be developed for the staggering problem (i.e., the greater of $\sum_i q_i/2$ or $\max_i\{q_i\}$), but the branch-and-bound procedure quickly identifies superior bounds.

Thus, we turn our attention to reducing the problem size through symmetry reduction, as branch-and-bound procedures can become quite inefficient when the problem contains many symmetries (Bosch & Trick, 2005). The offsetting problem under consideration is highly symmetric. For example, an optimal solution to the Murthy et al. (2003) 9-item problem is such that the time of the first replenishment for each item is at time $j = [0,\ 0,\ 1,\ 3,\ 2,\ 2,\ 0,\ 9,\ 1]$. An alternate optimal solution would simply advance all of these times by one period; that is, $j = [1,\ 1,\ 2,\ 4,\ 3,\ 3,\ 1,\ 10,\ 0]$ (see Proposition 2 of Yao et al. (2008)). These symmetries are problematic in a branch-and-bound process, since they increase the size of the search space and, perhaps even worse, result in wasted time searching the branch-and-bound tree that are symmetric to already visited—and failed—states (Walsh, 2012). Symmetry-breaking constraints can be included in the mixed-integer program to assist in eliminating this type of symmetry. In our implementation, we will "fix" some variables by including constraints that ensure the first replenishment of an item is at time 0 (i.e., for item i, $X_{i0} = 1$ and $X_{ij} = 0$ for $j = 1, \ldots, k_i - 1$).

As indicated by Yao et al. (2008), we can fix the variables for one item without loss of generality. To identify which item(s) should be selected for assignment, we consider two guidelines: (1) fix the item with the largest k_i, since the number of binary variables and the number of feasible solutions are dependent on these values, and (2) using the concept from the first-fit decreasing algorithm of bin packing, first assign the largest items (i.e., the largest q_i) placing the smaller items into the residual space (Constraint 3). To incorporate both guidelines, we propose fixing the item with the largest $k_i \times q_i$ value.

While previous research has suggested fixing only one item, we propose fixing additional items without compromising the optimal solution for situations in which the length of the planning horizon is greater than or equal to the least common multiple of the order cycle lengths of all items. The Chinese remainder theorem (e.g., van Tilborg, 2011) guarantees that, for two items with coprime order