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Decision Support

Quantifiers induced by subjective expected value of sample information with Bernstein polynomials

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ABSTRACT

A kind of personalized quantifier, the so-called SEVSI-induced quantifier as an acronym for Subjective Expected Value of Sample Information, is developed in this paper by introducing Bernstein polynomials of higher degree. This allows us to provide a novel solution to improve the final representation of the quantifier that generally performed poorly in our previous work, thus enhancing the quality of global approximation of functions and improving the operability of this kind of quantifier for practical use. We show some properties of the developed quantifier. We also prove the consistency of the OWA aggregation under the guidance of this type of quantifier. Finally, we experimentally show that the developed quantifier outperforms the one with the piecewise linear interpolation in many aspects of geometrical characteristics and operability. Thus it could be considered as an effective analytical tool to help handle the complex cases involving people's personalities or behavior intentions that have to be considered in decision making under uncertainty.

1. Introduction

In natural language there are many linguistic quantifiers exemplified by terms such as *more than 10*, *most*, *some*, *few*, and *about half*. So far there have been several attempts to deal with this topic, notable among these are the work by Liu (2005), Liu and Han (2008), Yager (2004, 1996), and Zadeh (1983, 1965). And a detailed overview can be found in Guo (2014). This paper is a direct continuation of our previous work (Guo, 2014) in which we proposed a kind of personalized quantifier, the so-called SEVSI-induced one as an acronym for Subjective Expected Value of Sample Information, which can be associated directly with a specified decision maker (DM) and used as a tool to investigate and formalize his/her decision attitude or behavior intention. This makes us believe that it has a wide range of applications in increasingly complex situations. In particular, it may help to bring about more intuitively appealing and convincing results in decision making under uncertainty, especially in the ordered weighted averaging (OWA) aggregation under the guidance of quantifier. Noted that the quantifier was realized in Guo (2014) by the piecewise linear interpolation and represented as a piecewise linear function, of which the number of pieces depends on the number of the alternatives in the given sample. Theoretically, the more pieces of the piecewise function, the better performance of the global approximation by this interpolation. For practical applications, however, too many pieces may throw users into confusion and such piecewise functions may become difficult to handle. Besides, the smoothness of the fitted curves by this interpolation may still need improving. Hence, more efforts should be made to tackle these issues so as to further perfect this kind of quantifier.

In this paper, we develop a new type of SEVSI-induced quantifier by introducing Bernstein polynomials of higher degree, thus providing a novel solution to the unsolved problems mentioned above. The consistency of the OWA aggregation under the guidance of the developed quantifier is also addressed and proved. Our aim is to develop a kind of personalized quantifier with excellent properties and pleasing operability, so as to provide an effective analytical tool with a sound theoretical basis for practical applications in more complex situations. Since the Regular Increasing Monotone (RIM) quantifier is a basis for constructing another two kinds of relative quantifiers, namely the Regular Decreasing Monotone (RDM) quantifier and the Regular UniModal (RUM) quantifier (Yager, 1996), all of the quantifiers considered in this paper are assumed to be RIM.

The rest of this paper is organized as follows: Section 2 briefly recalls the OWA operator and Bernstein polynomials. In Section 3, the SEVSI-induced quantifier with Bernstein polynomials of higher degree is investigated in considerable detail. Section 4 makes use of two numerical examples to illustrate and examine the developed quantifier, followed by conclusions in Section 5.

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2. Preliminaries

2.1. OWA operators

Yager (1988) introduced the concept of the OWA operator that is defined as follows.

The OWA operator of dimension n is a mapping $F : R^n \to R$ with an associated weighting vector $W = (w_1, w_2, \ldots, w_n)^T$ such that

$$F_W(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} w_j y_j, \tag{1}$$

where $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, and y_j is the jth largest of $x_i (i = 1, 2, \ldots, n)$. With the help of vector notations, Eq. (1) can be expressed as

$$F_W(x_1, x_2, \ldots, x_n) = W^T Y,$$

where $Y = (y_1, y_2, \ldots, y_n)^T$ is called an OWA argument vector.

It is well known that an OWA weighting vector plays a key role in the aggregation process (Ahn, 2011, Bustincea, Fernandeza, Kolesárovváb, & Mesiar, 2015). Yager (2009), Yager (2004), Yager (1996) suggested an effective approach to obtain an OWA weighting vector via the RIM quantifiers that can be denoted by a fuzzy subset Q with the following properties: 1) $Q(0) = 0$, 2) $Q(1) = 1$, 3)$Q(x) \geq Q(y)$if $x \geq y$. These quantifiers were denoted as basic unit-interval monotonic (BUM) functions in Yager (2004). Using a quantifier Q, we can obtain the OWA weights as

$$w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right), \quad j = 1, 2, \ldots, n \tag{2}$$

It is clear that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Eq. (1) can then be rewritten as

$$\begin{aligned} F_Q(x_1, x_2, \ldots, x_n) &= F_W(x_1, x_2, \ldots, x_n) \\ &= \sum_{j=1}^{n} \left[Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right) \right] y_j. \end{aligned} \tag{3}$$

This is called the quantifier guided OWA aggregation (Yager, 1996).

Yager (1993), Yager (1988) further introduced a characterizing measure called the attitudinal character that can be used to characterize an OWA weighting vector with respect to any distinction between preferences for large or small argument values. The attitudinal character is defined as

$$AC(W) = \sum_{j=1}^{n} w_j \frac{n-j}{n-1}. \tag{4}$$

It can be shown that $AC(W) \in [0, 1]$. Specifically, larger values of $AC(W)$, closer to 1, are an indication of preference for larger argument values in the aggregation. While lower values of $AC(W)$, closer to 0, are an indication of preference for smaller argument values in the aggregation. Values of $AC(W)$ in the middle, near 0.5, can be a reflection of no preference for either large or small argument values at all. Given a connection between Eq. (2) and Eq. (4), the measure of attitudinal character can be rewritten as

$$AC(Q) = \sum_{j=1}^{n} \left[Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right) \right] \frac{n-j}{n-1}. \tag{5}$$

Further algebraic manipulation of the formula leads to the following simple form

$$AC(Q) = \frac{1}{n-1} \sum_{j=1}^{n-1} Q\left(\frac{j}{n}\right). \tag{6}$$

Obviously, if $n \to +\infty$, then the concept of attitudinal character can be associated directly with a quantifier Q (Yager, 2004). In this case, let

$$\lambda_Q = \lim_{n\to\infty} AC(Q) = \int_0^1 Q(x)dx. \tag{7}$$

2.2. Bernstein polynomials

Recall the concept of classical Bernstein polynomials that is defined as follows.

Let $f(x) \in C[0, 1]$. A sequence of Bernstein polynomials of f is (Cheney, 1982)

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}, \tag{8}$$

where the binomial coefficients are given by $C_n^k = \frac{n!}{k!(n-k)!}$, and the Bernstein basic functions are defined by $B_n^k(x) = C_n^k x^k (1-x)^{n-k} (k = 0, 1, \ldots, n)$. According to Bernstein's proof of the Weierstrass Approximation Theorem, $\lim_{n\to\infty} B_n(f; x) = f(x)$ for any function $f(x) \in C[0, 1]$ (Cheney, 1982).

Bernstein polynomials have some best properties among all approximating polynomials (Ghosh & Osman, 2012), some of which are briefly introduced as follows.

1) Normalization of the Bernstein basic functions, i.e., $\sum_{k=0}^{n} B_n^k(x) = 1$ where $B_n^k(x) \geq 0$ for any $x \in [0, 1]$.

Proof. Trivial from the binomial expansion. □

2) End-point interpolation, i.e., $B_n(f; 0) = f(0)$, and $B_n(f; 1) = f(1)$.

Proof. Trivial from the algebraic manipulation of Eq. (8). □

3) Positive linear operator, i.e., for any $f(x), g(x) \in C[0, 1]$, $B_n(f + g; x) = B_n(f; x) + B_n(g; x)$, $B_n(af; x) = aB_n(f; x)$ where $a \in R$, and $B_n(f; x) \geq 0$ if $f(x) \geq 0$.

Proof. Trivial from the algebraic manipulation of Eq. (8). □

Thus it can be easily deduced that $B_n(f; x) \geq B_n(g; x)$ if $f(x) \geq g(x)$ for any $x \in [0, 1]$.

4) Linear invariant, i.e., for any linear function $l(x) = ax + b$ where $a, b \in R$, $B_n(l; x) = l(x)$.

Proof. It is clear from the property 3) that $B_n(ax + b; x) = aB_n(x; x) + bB_n(1; x)$, where

$$B_n(1; x) = \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = (x + (1-x))^n = 1,$$

and

$$\begin{aligned} B_n(x; x) &= \sum_{k=0}^{n} \frac{k}{n} C_n^k x^k (1-x)^{n-k} = \sum_{k=1}^{n} \frac{k}{n} C_n^k x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n-1} \frac{k+1}{n} C_n^{k+1} x^{k+1} (1-x)^{n-k-1} \\ &= x \sum_{k=0}^{n-1} C_{n-1}^k x^k (1-x)^{n-k-1} = x(x + 1 - x)^{n-1} = x. \end{aligned}$$

Thus $B_n(ax + b; x) = aB_n(x; x) + bB_n(1; x) = ax + b$, i.e., $B_n(l; x) = l(x)$. □

This means all of the linear functions are the fixed points of B_n.

5) Geometrically shape-preserving property, i.e., $B_n(f; x)$ is monotonically increasing (or convex) over $[0, 1]$ if the function $f(x) \in C[0, 1]$ does so.

Proof. (a) Assume that $f(x) \in C[0, 1]$ is monotonically increasing (or decreasing) over $[0, 1]$. We shall prove $B_n(f; x)$ does so by