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# Age-structured linear-state differential games

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ABSTRACT

In this paper we search for conditions on age-structured differential games to make their analysis more tractable. We focus on a class of age-structured differential games which show the features of ordinary linear-state differential games, and we prove that their open-loop Nash equilibria are sub-game perfect. By means of a simple age-structured advertising problem, we provide an application of the theoretical results presented in the paper, and we show how to determine an open-loop Nash equilibrium.



## 1. Introduction

In this paper we want to extend the concept of linear-state differential game to a family of models with age-structured dynamics. In the last few years there has been an increasing number of papers on applications of age-structured optimal control. For an introduction to this topic the reader can consult the book by Grass, Caulkins, Feichtinger, Tragler, and Behrens (2008, Chapter 8, pp. 417–421) or the book by Aniţa, Arnăutu, and Capasso (2011, Chapter 4, pp. 145–184). Moreover, three papers are important for our analysis. Feichtinger, Tragler, and Veliov (2003) introduce a very general set of Pontryagin-type necessary conditions, while Feichtinger, Hartl, Kort, and Veliov (2006) describe the anticipation effect: in age-structured models it is convenient to anticipate an investment flow to take advantage of the age evolution. Finally, Krastev (2013) presents a set of Arrow-type sufficient conditions. This is definitely a non-exhaustive list, but browsing the references in the just mentioned papers the reader can find a variety of different applications in this active research field. On the other hand, only a few applications of these mathematical techniques to differential games have been proposed so far. To the best of our knowledge, one of the first papers on applications of partial differential games is by Roxin (1977), who presents two examples: one on pollution control, the other on competitive fishing. A more technical paper on partial differential games is by Ichikawa (1976), who shows how to study the linear-quadratic differential games when the motion equation is described using a strongly continuous semigroup. Even if this approach is rather technical, the reference may be useful for further research on age-structured differential games. Finally, as far as we know, the most recent application is by Reluga and Li (2013) in the field of mathematical biology.

The fact that there are only few applications of age-structured optimal control to differential games is surely due to the complexity that arises in the analysis of the necessary conditions for this kind of models. However, even ordinary differential games are difficult to analyze, so that, in the applications, it is customary to focus on some special families of differential games. Two of them are the *linear-quadratic differential games* Engwerda (2005) and the *linear-state differential games* (Dockner, Jørgensen, Van Long, and Sorger, 2000, Chapter 7, p. 187). The analysis of these kinds of differential games is mathematically tractable and allows the characterization of strong equilibria: for example it has been proved that an open-loop Nash equilibrium in a linear-state differential game is sub-game perfect (Dockner, Jørgensen, Van Long, and Sorger, 2000, Chapter 7, p. 189).

In this paper we define a class of age-structured differential games which has the features of ordinary linear-state differential games. In Section 2 we present a quick review of age-structured optimal control and we introduce a formulation of an age-structured differential game. In Section 3 we introduce the linear-state formulation of an age-structured differential game and we show that it is the right formulation, because we can prove the sub-game perfectness of the open-loop Nash equilibria. In Section 4 we describe a very simple advertising model with age-structured dynamics which is useful to show how to characterize an open-loop Nash equilibrium.

## 2. Age-structured models

### 2.1. Age-structured optimal control problems

First of all we present a formal definition of an age-structured optimal control problem. We inform the readers that the material

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introduced in this subsection is presented to make the paper self-contained and is taken from the seminal work by Feichtinger et al. (2003).

**Definition 1.** An age-structured optimal control problem is defined by stating an objective functional to maximize

$$\max_{u(t,a)\in U, v(t)\in V} \Big\{ J(u(t,a), v(t)) = \int_0^T \int_0^\omega L(t,a,y(t,a),p(t,a),u(t,a))\,da\,dt + \int_0^\omega \ell(a,y(T,a))\,da + \int_0^T \mathcal{L}(t,q(t),v(t))\,dt \Big\}, \tag{1}$$

subject to a motion PDE

$$\begin{aligned}(\partial_t + \partial_a) y(t,a) &= f(t,a,y(t,a),p(t,a),q(t),u(t,a)),\\ y(0,a) &= y_0(a),\\ y(t,0) &= k(t,q(t),v(t)),\end{aligned} \tag{2}$$

where the non-local variables $p(t, a)$ and $q(t)$ are defined as follows:

$$\begin{aligned} p(t,a) &= \int_0^\omega g(t,a,\sigma,y(t,\sigma),u(t,\sigma))\,d\sigma,\\ q(t) &= \int_0^\omega h(t,\sigma,y(t,\sigma),u(t,\sigma))\,d\sigma. \end{aligned} \tag{3}$$

A feasible control $u(t, a)$ is an element of the set of the measurable and essentially bounded functions $L_\infty([0, T] \times [0, \omega]; U)$ where $U$ is a compact subset of $\mathbb{R}$; a feasible control $v(t)$ belongs to $L_\infty([0, T]; V)$ where $V$ is a compact and convex subset of $\mathbb{R}$. Given a couple of feasible controls $(u(t,a), v(t))$ there exists a unique state function $y(t,a) \in L_\infty([0,T]\times[0,\omega];\mathbb{R})$ which satisfies (2) and (3). The couple of feasible controls $(u^*(t,a), v^*(t))$ is optimal if and only if

$$J(u^*(t,a), v^*(t,a)) \geq J(u(t,a), v(t,a))$$

for all couples of feasible controls $(u(t,a), v(t,a))$.

In this definition we are assuming that the functions $L, \ell, \mathcal{L}, f, g, h, k$ are measurable with respect to the variables $t$, $a$, $\sigma$, continuous with respect to the remaining variables, locally essentially bounded, and differentiable w.r.t. $y, p, q, u, v$. Moreover, the partial derivatives are measurable w.r.t. $t$, $a$, $\sigma$, continuous w.r.t. the remaining variables, and locally essentially bounded. For further details on the analytical setting we refer to the seminal paper by Feichtinger et al. (2003) where the Pontryagin-like necessary conditions are introduced. The idea behind this formulation is the following: the decision-maker wants to control the evolution of the PDE in the set $[0, T] \times [0, \omega]$; the first variable $t \in [0, T]$ represents the time (the programming horizon is finite, $T > 0$), while the second variable $a \in [0, \omega]$ represents the age ($\omega > 0$ is the maximum age we take into account). The quantity $y(t, a)$ is the value of the state function at the time $t$ for the "class" of age $a$. Time and age evolve together; the motion equation describing the state variable evolution in time and age is a linear PDE. In the two segments $\{0\} \times [0, \omega]$ and $[0, T] \times \{0\}$ we define the boundary conditions as follows: at the time $t = 0$ the value of the state function is known for all $a \in [0, \omega]$ and it is given by the function $y_0(a)$; while at each time $t$ the value of the state function for the age $a = 0$ is given in an implicit way. This value is defined using the equation $y(t,0) = k(t,q(t),v(t))$ where $v(t)$ is a control, while $q(t)$ is a "non-local" variable that depends on the shapes of the state and control functions along the segment $\{t\} \times (0, \omega]$. A clarifying example is provided in the field of population dynamics, where $y(t, a)$ represents the number of people of age $a$ at the time $t$. In this kind of model the number of newborns at a given time (i.e. $y(t, 0)$) depends on the age distribution of the population at that time (to explore this topic we suggest the interesting paper written by Simon, Skritek, and Veliov, 2013 and the references therein). The non-local variable $q(t)$ is present also in the motion equation. In order to describe a different phenomenon another non-local variable $p(t, a)$ is defined. This variable represents the influence of the "age-class" $\sigma$ on the "age-class" $a$ at a given fixed time $t$. In our opinion, an application that clearly explains the meaning of this non-local variable in the context of drug initiation is by Almeder, Caulkins, Feichtinger, and Tragler (2004). In this model $y(t, a)$ represents the number of non-drug-users. At a fixed time $t$, for all $\sigma \in [0, \omega]$, the number of non-drug-users in the age-class $\sigma$ (i.e. $y(t, \sigma)$) can influence the evolution of $y(t, a)$ because of reputation interactions among different age groups.

The decision maker chooses the control functions $u(t, a)$ and $v(t)$ in order to maximize the objective functional (1). This is the sum of three terms: the first one depends on the values of the state and control functions on their whole domain $[0, T] \times [0, \omega]$; the second term depends on the values of the state function in the final segment $\{T\} \times [0, \omega]$; finally, the third term depends on the values of the state and control functions through the synthesis along the segment $\{t\} \times (0, \omega]$ provided by the function $q(t)$.

The formulation of the problem in Definition 1 is less general than other formulations considered in literature, as we assume that the initial value for the state function $y(t, a)$ is given a priori (it does not depend on further control), and that the quantity $q(t)$ is defined explicitly ( the function $h$ does not depend on $p$ or $q$). This clarifies the presentation and makes the analytical setting simpler.

Given the optimal control problem introduced in Definition 1, we define the distributed Hamiltonian

$$\begin{aligned} &H(t,a,y,p,q,u,\lambda,\eta(t,\sigma),\zeta)\\ &= L(t,a,y,p,q,u) + \lambda f(t,a,y,p,q,u)\\ &\quad + \int_0^\omega \eta(t,\sigma) g(t,a,\sigma,y,u)\,d\sigma + \zeta h(t,a,y,u) \end{aligned} \tag{4}$$

and the boundary Hamiltonian

$$H^b(t,q,v,\lambda(t,0)) = \mathcal{L}(t,q,v) + \lambda(t,0) k(t,q,v)\,. \tag{5}$$

Moreover, we define the adjoint variables as follows: $\lambda(t, a)$ satisfies

$$\begin{aligned} (\partial_t + \partial_a)\lambda(t,a) &= -\partial_y H(t,a,y(t,a),p(t,a),q(t),u(t,a),\\ &\qquad \lambda(t,a),\eta(t,\sigma),\zeta(t)),\\ \lambda(t,\omega) &= 0,\\ \lambda(T,a) &= \partial_y \ell(a,y(T,a)), \end{aligned} \tag{6}$$

while $\eta(t, a)$ and $\zeta(t)$ are defined explicitly by the equations:

$$\begin{aligned} \eta(t,a) &= \partial_p H(t,a,y(t,a),p(t,a),q(t),u(t,a),\\ &\qquad \lambda(t,a),\eta(t,a),\zeta(t)),\\ \zeta(t) &= \partial_q H^b(t,q(t),v(t),\lambda(t,0))\\ &\quad + \int_0^\omega \partial_q H(t,a,y(t,a),p(t,a),q(t),u(t,a),\\ &\qquad \lambda(t,a),\eta(t,a),\zeta(t))\,da. \end{aligned} \tag{7}$$

We observe that, under our assumptions, Eq. (7) defines the quantities $\eta(t, a)$ and $\zeta(t)$ explicitly, as the functions $\partial_q H$ and $\partial_p H$ do not depend on the two variables $\eta$ and $\zeta$. The assumptions introduced after Definition 1 allow us to apply the necessary conditions for optimality described by Feichtinger et al. (2003) and recalled by Krastev (2013).

**Theorem 1** (Necessary conditions)**.** *Let $(u^*(t,a), v^*(t))$ be an optimal couple of controls for the age-structured problem (1)–(3) and let $y^*(t, a)$ be the state function associated with that couple of controls.*