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Discrete Optimization

How bad can the centroid be?☆

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A R T I C L E   I N F O

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A B S T R A C T

In this note we first show that the centroid (or centre of gravity) gives in value a $(\sigma + 1)$-approximation to any continuous single facility minisum location problem for any gauge with asymmetry measure σ, and thus a 2-approximate solution for any norm.

On the other hand for any gauge the true minimum point (the 1-median) remains within a bounded set whenever a fixed proportion of less than half of the total weight of the destination points is moved to any other positions. It follows that the distance between the centroid and the 1-median may be arbitrary close to half the diameter of the destination set.

1. Introduction

The Fermat–Weber problem, or single facility minisum Euclidean location problem, is probably the most studied and discussed continuous location model, see e.g. the survey (Drezner, Klamroth, Schöbel, & Wesolowsky, 2003). In its modern form it is stated as follows:

Given a finite set of destination points $A \subset \mathbb{R}^d$ and weights $w_a > 0$ $(a \in A)$, find the point x minimizing the sum of weighted (Euclidean) distances to all points of A, i.e. solve

$$\min \left\{ \sum_{a \in A} w_a \|a - x\| \mid x \in \mathbb{R}^d \right\} \tag{1}$$

where we assume for notational simplicity and without loss of generality that $\sum_{a \in A} w_a = 1$. Such a point is also called a Weber-point or (Euclidean) 1-median.

It was shown in Bajaj (1988) that no closed algebraic form can solve it in general. Therefore it should be solved through iterative techniques from convex optimisation, the most popular for this particular problem being the almost centenary method developed by Weiszfeld (Weiszfeld, 1937; Weiszfeld & Plastria, 2009), as explained in detail in Plastria (2011).

In 1937, Keefer (according to Eilon, Watson-Gandy, and Christofides (1971)—we were unable to verify this information) introduced the so-called *centroid* (or centre of gravity) *method*, stating that the solution would be

$$g = \sum_{a \in A} w_a a \tag{2}$$

Since then, till today, many books on Operations Management, including e.g. Weida, Richardson, and Vazsonyi (2001) (note that Vazsonyi=Weiszfeld, see Vazsonyi, 2002) propose this centroid solution as optimal. This is totally wrong. The centroid g minimizes another objective: the sum of weighted *squared* Euclidean distances, as is easily established, see e.g. Plastria (2011). Attempts to banish this error have been numerous, but seem not to have been sufficiently heard and still continue to be felt necessary, as exemplified by the writings of Eilon et al. (1971); Schärlig (1973); Vergin and Rogers (1967), and others, up to much more recently (Gehrlein & Pasic, 2009).

The difficulty to convince the operations management community of this error should probably be sought for a good part in the fact that in simple examples the centroid seems to give a quite satisfactory approximation of the optimal solution, and, being so much easier to calculate, is therefore preferred. This argument is for example central to the recent paper (Kuo, 2010) that advocates the centroid as a good heuristic solution, and attempts to show so experimentally using randomly generated data, concluding that the approximation grows better and better with increasing number of destinations. This is however no proof of goodness in general at all. In fact it only illustrates the following rather evident fact about the quite exceptional case where the destinations admit a symmetry centre.

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Lemma 1. *If the destinations and their weights admit a symmetry centre, i.e. there exists a point s such that for each $a \in A$ the symmetric point $a^s = s + (s - a) \in A$ and has the same weight ($w_a = w_{a^s}$), then s is at the same time an optimal solution to (1) and the centre of gravity g, as well as being a point minimizing the weighted sum of any fixed power (≥ 1) of distances to the destinations.*

Proof. For any $p \geq 1$ the function $f^p(x) = \sum_{a \in A} w_a \|a - x\|^p$ is a convex function of x. Thanks to the assumed existence of s and the symmetry of the Euclidean norm we also have the symmetry property that for any $x \in \mathbb{R}^d$ we have

$$f^p(x^s) = f^p(s + (s - x)) \tag{3}$$

$$= \sum_{a \in A} w_a \|a - (s + (s - x))\|^p \tag{4}$$

$$= \sum_{a \in A} w_a \|x - (s + (s - a))\|^p \tag{5}$$

$$= \sum_{a \in A} w_{a^s} \|a^s - x\|^p \tag{6}$$

$$= \sum_{a^s \in A} w_{a^s} \|a^s - x\|^p \tag{7}$$

$$= f^p(x) \tag{8}$$

But $s = 0.5x + 0.5x^s$ for any x, so by convexity $f^p(s) \leq 0.5 f^p(x) + 0.5 f^p(x^s) = f^p(x)$, showing that s minimizes f^p.

Taking $p = 1$ we obtain that s is an optimal solution to (1), and taking $p = 2$ we obtain $s = g$ (which, evidently, may be obtained more directly by $2g = \sum_{a \in A} w_a a + \sum_{a^s \in A} w_{a^s} a^s = \sum_{a \in A} w_a (a + a^s) = 2s$). □

Thus, when the random generation of coordinates and weights is done using a uniform distribution (as is not said, but implicit in Kuo (2010)) the sample test data obtained approximates a symmetric destination set better and better as the sample size increases. And therefore it should be expected that both g and the Weber point will converge to the same asymptotic symmetry centre of the destinations.

So, let us have a deeper look at the question how bad the centroid can be for 'solving' single facility minisum location problems. As Euclidean distance is a rather restrictive view on modelling real world distances, see e.g. Plastria (1995), we develop our analysis in the much more general setting where distance is measured by an arbitrary finite gauge, the generalisation of a norm that includes possible asymmetry.

2. Comparison of the 1-median and centroid in value

A first and most common way to check the fit of a solution is to compare it in value to the optimal.

Given any gauge ν on $\mathbb{R}^d$ we consider the following single facility minisum location problem (see e.g. Plastria, 2009 for general properties of norms and gauges).

$$\min \left\{ f(x) = \sum_{a \in A} w_a \nu(a - x) \mid x \in \mathbb{R}^d \right\} \tag{9}$$

and call any optimal solution m a 1-median. It is well known (see e.g. Pelegrin, Michelot, & F.Plastria, 1985) that multiple optimal solutions might exist, but only when A is aligned and/or ν is not round (i.e. its unit ball is not strictly convex).

We investigate the difference between $f(g)$ and $f(m)$.

Recall that for the gauge ν we have the triangle inequality, in particular $\nu(w) - \nu(v) \leq \nu(w - v)$ for any $v, w \in \mathbb{R}^d$. Due to possible asymmetry of ν the left-hand side of this inequality may not be inversed. Using the skewness σ of the gauge ν, introduced in Plastria (2009) as

$$\sigma \stackrel{\text{def}}{=} \max\{\nu(-x) \mid \nu(x) = 1\} \tag{10}$$

we have $\nu(-y) \leq \sigma \nu(y)$ for any $y \in \mathbb{R}^d$ and so

$$\forall v, w \in \mathbb{R}^d : |\nu(v) - \nu(w)| \leq \sigma \nu(w - v). \tag{11}$$

Note that $\sigma \geq 1$, with equality only in case of symmetry, so when ν is a norm. The easiest example of a gauge with skewness $\sigma > 1$ is the one-dimensional gauge ν_σ defined on $\mathbb{R}$ as

$$\nu_\sigma(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{when } x \geq 0 \\ -\sigma x & \text{when } x < 0 \end{cases} \tag{12}$$

Equality in the extended triangle inequality (11) is only possible when v, w and $w - v$ lie on a same (flat) face of ν's unit ball B_ν, meaning either that $v = w$ or that ν is not round: there exists some $p \neq 0$ such that $\nu(v) = \langle p\ ;\ v\rangle$, $\nu(w) = \langle p\ ;\ w\rangle$ and $\nu(w - v) = \langle p\ ;\ w - v\rangle$.

Lemma 2 (Lipschitz property)**.** *For all $x, y \in \mathbb{R}^d$ we have $|f(y) - f(x)| \leq \sigma \nu(y - x)$*

Proof.

$$|f(y) - f(x)| = \left| \sum_{a \in A} w_a (\nu(a - y) - \nu(a - x)) \right| \tag{13}$$

$$\leq \sum_{a \in A} w_a |\nu(a - y) - \nu(a - x)| \tag{14}$$

$$\text{by (11)} \leq \sum_{a \in A} w_a \sigma \nu(y - x) \tag{15}$$

$$= \sigma \nu(y - x) \tag{16}$$

We therefore have

Theorem 3. $f(m) \leq f(g) \leq (\sigma + 1) f(m)$

Proof. The first inequality holds because m is a minimum of f.

Using Lemma 2 for $y = g = \sum_{a \in A} w_a a$ we obtain for $x = m$

$$f(g) - f(m) \leq \sigma \nu(g - m) \tag{17}$$

$$= \sigma \nu \left(\sum_{a \in A} w_a a - m \right) \tag{18}$$

$$= \sigma \nu \left(\sum_{a \in A} w_a (a - m) \right) \tag{19}$$

$$\leq \sigma \sum_{a \in A} w_a \nu(a - m) = \sigma f(m) \tag{20}$$

It follows that $f(g) \leq (\sigma + 1) f(m)$. □

The next question is now : can this upper bound of $(\sigma + 1)$ on the approximation factor be reached?

Theorem 4. *If there are at least two different $a \in A$ we always have $f(g) < (\sigma + 1) f(m)$*

Proof. For an equality $f(g) = (\sigma + 1) f(m)$ we would need equalities instead of inequalities everywhere in the proof of Theorem 3.

Now by convexity of ν, inequality (20) only holds if ν is linear on the convex hull of the $a - m$ ($a \in A$). This would mean that there exists a $p \neq 0$ such that for all $a \in A$

$$\nu(a - m) = \langle p;\ a - m\rangle \tag{21}$$