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Decision Support

Random-payoff two-person zero-sum game with joint chance constraints

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ABSTRACT

We study a two-person zero-sum game where the payoff matrix entries are random and the constraints are satisfied jointly with a given probability. We prove that for the general random-payoff zero-sum game there exists a "weak duality" between the two formulations, i.e., the optimal value of the minimizing player is an upper bound of the one of the maximizing player. Under certain assumptions, we show that there also exists a "strong duality" where their optimal values are equal. Moreover, we develop two approximation methods to solve the game problem when the payoff matrix entries are independent and normally distributed. Finally, numerical examples are given to illustrate the performances of the proposed approaches.

1. Introduction

In this paper, we consider the classic two-person zero-sum game, i.e., there are only two players where one player wins what the other player loses. Without loss of generality, we refer to the players as Player I and Player II. Let $A = (a_{i,j})_{n\times m}$ be the payoff matrix of a two-person zero-sum game. If Player I plays pure strategy i and Player II uses pure strategy j, then the payoff from Player II to Player I is a_{ij}. In the game, Player I seeks a mixed strategy to maximize his minimum payoff while Player II seeks a mixed strategy to minimize his maximum loss. In this case, the two-person zero-sum game can be mathematically formulated as two linear programming (LP) problems:

For Player I,

$$\begin{aligned} d^* := \max\ & d \\ s.t.\ & \min_y (x^T A y : y^T e_m = 1, y \geq 0) \geq d \\ & x^T e_n = 1, x \geq 0 \end{aligned} \tag{1}$$

or

$$(P1)\quad \begin{aligned} d^* := \max\ & d \\ s.t.\ & A^T x \geq d e_m \\ & x^T e_n = 1, x \geq 0 \end{aligned} \tag{2}$$

where e_k is a $k-$dimensional vector with all elements are equal to 1.

For Player II,

$$\begin{aligned} t^* := \min\ & t \\ s.t.\ & \max_x (x^T A y : x^T e_n = 1, x \geq 0) \leq t \\ & y^T e_m = 1, x \geq 0 \end{aligned} \tag{3}$$

or

$$(P2)\quad \begin{aligned} t^* := \min\ & t \\ s.t.\ & Ay \leq t e_n \\ & y^T e_m = 1, y \geq 0 \end{aligned} \tag{4}$$

With a deterministic payoff matrix A, the famous von Neumann minimax theorem von Neumann (1928) states that the two optimal values of the two LP problems are equal, i.e., $d^* = t^*$. However, in practice, due to modeling or prediction errors, different kinds of uncertainties occur. Therefore, the payoff matrix is unknown in advance. In this case, it is natural to model the payoff matrix by continuously or discretely distributed random variables, which turns the underlying problem into a stochastic optimization problem. If the random payoff matrix is replaced by its expectation, then the variability of the payoffs is not captured. In this paper, we consider chance-constrained criteria for determining optimal strategies Charnes, Kirby, and Raike (1968). Each player optimizes his strategy and return such that the probability of attaining that return is at least some given value. In this context, the random-payoff two-person zero-sum game can be formulated as the following stochastic programming problems:

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For Player I,

$$(P3): \quad P^* := \max_{x,d} \ d$$
$$s.t. \ \ \boldsymbol{Pr}\{A^T x \geq d e_m\} \geq \alpha$$
$$x^T e_n = 1, x \geq 0 \qquad (5)$$

For Player II,

$$(P4): \quad D^* := \min_{y,t} \ t$$
$$s.t. \ \ \boldsymbol{Pr}\{Ay \leq t e_n\} \geq \beta$$
$$y^T e_m = 1, y \geq 0 \qquad (6)$$

The constraints (5) and (6) are called joint chance constraints. For the sake of simplicity, we call the game a chance-constrained game hereafter.

The rest of the paper is organized as follows. In Section 2, we review the literature related to chance-constrained games. In Sections 3 and 4, we investigate the relationship between the best payoff of the two players, and derive both weak and strong duality results. In Section 5, we consider the special case where the payoff matrix entries are independent and normally distributed. Two approximation methods to solve the problem are given, namely a conservative approximation and the other is a relaxed approximation. A numerical study is given in Section 6. Finally, the last section contains the conclusions.

2. Literature review

Game theory, as a mathematical study on conflict and cooperation situations, has been widely studied in several fields, namely economics, networks, political science and psychology and recently computer science. Since von Neumann and Morgenstern (1947) developed game theory, this topic has attracted the attention of economists, mathematicians and operations researchers. We refer the reader to some famous game theory textbooks Dixit, Skeath, and Reiley (2014); Gibbons (1992).

Nash (1950) proved that there exists at least one Nash equilibrium where no player can improve his expected payoff by changing his strategy. Peski (2008) presented simple necessary and sufficient conditions for the comparison of information structures in zero-sum games. In game theory, two-person zero-sum game is one of the fundamental problems with two players where one player wins what the other one loses. When the payoff matrix is deterministic, von Neumann (1928) proved that the best payoff of one player is equal to the best one of the other player. However, in practice, due to modeling or prediction errors, the payoff matrix is not known in advance. Stochasticity in game theory can be handled by at least two different approaches: stochastic optimization Blau (1974); Cassidy, Field, and Kirby (1972); Charnes et al. (1968) which considers payoffs as random variables, and fuzzy sets approach Aubin (1981); Butnariu (1978); Xu, Zhao, and Ning (2006).

Flesch, Schoenmakers, and Vrieze (2009) studied the product-game where there are $n-$ player stochastic games played on a product state space. They establish the existence of $0-$ equilibria. In the case of two-players zero-sum games of this type, they show that both players have stationary $0-$optimal strategies. Product-games with an aperiodic transition structure were also considered in Flesch, Schoenmakers, and Vrieze (2008). Monroy, Hinojosa, Marmol, and Fernandez (2013) considered stochastic cooperative games where the coalition values are fuzzy. Stochastic linear programming games were studied by Ulhan (2015) where the uncertainty of the payoffs is determined by a specially structured linear program. When each player preferences over random payoffs are represented by a concave functional, the author proves that these games have a nonempty core. A computable algorithm to calculate uniform $\epsilon-$optimal strategies in two-players zero-sum stochastic games was proposed in Solan and Vieille (2010). This approach can be used to construct $\epsilon-$equilibria algorithms in various classes of multi-player non-zero-sum stochastic games.

Duality in linear programming can be formulated as a game theory problem. In Bot, Lorenz, and Wanka (2010), the authors give the deterministic equivalent formulation of a linear chance-constrained optimization problem and its conjugate dual problem. They provide weak sufficient conditions which ensure strong duality for this primal-dual pair. Komaromi (1992) proposes two chance constrained linear programming problems pairs. He provides conditions under which the dually related problems have no duality gap.

In this paper, we investigate two-person zero-sum game problem with a random payoff matrix. There are different ways to deal with the uncertainty. For instance, if we replace the random payoff matrix by its expectation, stochastic variability and risk measure will not be considered, see Song (1992) for more details. Charnes et al. (1968) first introduced the chance constrained programming method to address random payoffs in the two-person zero-sum game problem. Later on, Song (1992) extended the major results of Charnes et al. (1968) for joint chance constraints. The main idea of his conservative method is to introduce the quantile of each element of the matrix such that the chance constraints are satisfied. In this paper, we study the relationship between the best payoff of the two players under joint chance constraints. Precisely, we prove that there exists a "weak duality" between the two formulations, i.e., the optimal value of player I is an upper bound of the one of player II. Moreover, we show that there exists a "strong duality" under certain assumptions. Additionally, when the payoff matrix entries are independent and normally distributed, two approximation methods of the game problem are given, namely a conservative approximation and a relaxed approximation. We call an approximation hereafter conservative when its feasible set is a subset of the feasible set of the original problem whilst it is relaxed when the feasible set of the original problem is subset of the relaxed one. Finally, numerical tests on random generated instances are given to illustrate the quality of our approaches.

3. Stochastic dual problem

For the deterministic payoff, the two LP problems are a pair of primal and dual problems and their optimal values are the same. However, it is not the case any more for stochastic games. In this section, we investigate the relationship between the stochastic primal and dual problems.

Theorem 3.0.1. *(**"Weak Duality"**) Let P^* and D^* be the optimal values of Problems (5) and (6) respectively. If $\alpha > 0.5$ and $\beta > 0.5$, then $P^* \leq D^*$.*

Proof. We first consider the case when the payoff matrix A is discretely distributed. Without loss of generality, let $\boldsymbol{Pr}(A = A_i) = p_i, \ i = 1, \dots, N$.

Let (d^*, x^*) and (t^*, y^*) be the optimal solutions of the "primal" problem and the "dual" problem respectively. Then, there exists two subsets $J_1, J_2 \subset \{1, \dots, N\}$ such that $\sum_{j \in J_1} \boldsymbol{Pr}(A = A_j) \geq \alpha$ and $A_j^T x^* \geq d^* e_m, for j \in J_1$ while $\sum_{j \in J_2} \boldsymbol{Pr}(A = A_j) \geq \beta$ and $A_j y^* \leq t^* e_n$, $for j \in J_2$.

For $j \in J = J_1 \cup J_2$, we define problems (P_j) and (D_j) as follows:

$$(P_j): \quad P_j^* := \max \ d$$
$$s.t. \ \ A_j^T x \geq d e_m$$
$$x^T e_n = 1, x \geq 0$$