European Journal of Operational Research 000 (2015) 1–13

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Discrete Optimization

Integer linear programming models for the skiving stock problem

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ARTICLE INFO

Article history:
Received 18 January 2015
Accepted 3 November 2015
Available online xxx

Keywords:
Packing
Skiving stock problem
Dual bin packing
Modeling
Continuous relaxation

ABSTRACT

We consider the one-dimensional skiving stock problem which is strongly related to the dual bin packing problem: find the maximum number of items with minimum length L that can be constructed by connecting a given supply of $m \in \mathbb{N}$ smaller item lengths $l_1, \ldots, l_m$ with availabilities $b_1, \ldots, b_m$. For this optimization problem, we present three new models (the arcflow model, the onestick model, and a model of Kantorovich-type) and investigate their relationships, especially regarding their respective continuous relaxations. To this end, numerical computations are provided. As a main result, we prove the equivalence between the arcflow model, the onestick approach and the existing pattern-oriented standard model. In particular, this equivalence is shown to hold for the corresponding continuous relaxations, too.

1. Introduction

In this paper, we consider the one-dimensional skiving stock problem (SSP) which is strongly related to the dual bin packing problem (DBPP). In the classical formulation, $m \in \mathbb{N}$ different item lengths $l_1, \ldots, l_m$ with availabilities $b_1, \ldots, b_m$ are given, the so-called *item supply*. We aim at maximizing the number of products with minimum length L that can be constructed by connecting the items on hand. Such computations are of high interest in many real world applications, e.g. industrial production processes (see Zak, 2003 for an overview) or politico-economic problems (cf. Assmann, Johnson, Kleitman, & Leung, 1984; Labbé, Laporte, & Martello, 1995). Furthermore, also neighboring tasks, such as dual vector packing problems (Csirik, Frenk, Galambos, & Rinnooy Kan, 1991) or the maximum cardinality bin packing problem (Bruno & Downey, 1985; Peeters & Degraeve, 2006), are often associated or even identified with the dual bin packing problem. These formulations are of practical use as well since they are applied in multiprocessor scheduling problems (Alvim, Ribeiro, Glover, & Aloise, 2004) or surgical case plannings (Vijayakumar, Parikh, Scott, Barnes, & Gallimore, 2013).

The considered optimization problem, for the case $b_i = 1$ ($i \in I := \{1, \ldots, m\}$), was firstly mentioned by Assmann et al. (1984), based on the doctoral thesis (Assmann, 1983), and denoted by dual bin packing problem. Therein, the authors mainly investigate heuristic approaches and provide results regarding the quality and average case behavior of the presented methods. Further contributions, especially in terms of exact approaches to the DBPP, have been studied in Labbé et al. (1995) and Peeters and Degraeve (2006) where two branching algorithms are introduced.

Based on practical preliminary thoughts (Johnson, Rennick, & Zak, 1997), a generalization for larger availabilities $b_i \in \mathbb{N}$ ($i \in I$) has been considered in Zak (2003) also motivating the term of skiving stock problem. In that paper, Zak formulates a pattern-oriented model of the SSP with an infinite number of variables and provides first (numerical) results regarding the gap of this optimization problem, i.e., the difference between the optimal values of the continuous relaxation of the SSP and the SSP itself. But to our best knowledge, there are only a few works concerning theoretical aspects.

In this paper, we introduce three new models for the skiving stock problem which only require a pseudo-polynomial number of variables and constraints. Two of them are based on position-indexed and order-indexed approaches that are related to the arcflow and onecut formulations (see Dyckhoff, 1981; Valério de Carvalho, 2002) of the extensively studied one-dimensional cutting stock problem. In contrast to the standard model of Zak (2003), these new models can cope with larger and practical meaningful instances. Additionally, we provide some techniques in order to significantly reduce the total amount of variables and constraints within these two models. For the sake of completeness, the third model refers to the well known Kantorovich approach (Kantorovich, 1939) and is shown to have a weak continuous relaxation and many symmetric solutions. For this purpose, we present an approach to avoid symmetries with the help of additional linear constraints and binary variables.

We provide theoretical and numerical investigations on the relationship of these models and prove, as a main result, the equivalence

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http://dx.doi.org/10.1016/j.ejor.2015.11.005
0377-2217/

between the continuous relaxations of the standard model and the new position-indexed and order-indexed formulations.

Throughout this paper, we assume without loss of generality that all input-data are positive integers. Additionally, we expect the item lengths to be smaller than L since each item $i \in I$ with $l_i \geq L$ already represents a finished product and does not have to be considered within the optimization. Finally, we assume that two items can be connected directly, i.e., without any overlap or an intermediate composite. If, in a certain practical application, this situation is not given from the beginning, we can, depending on the particular case, shorten or extend the item lengths and L obtaining an equivalent skiving stock problem where this third assumption is fulfilled.

The next section deals with the above mentioned standard model (Zak, 2003). Section 3 introduces an arcflow and an onestick model, respectively, and proves their equivalence to Zak's formulation. Afterwards, we present a model of Kantorovich-type and show how symmetries can be avoided therein. In a final step, we compare all approaches by means of numerical computations, give some conclusions and an outlook of future research.

2. A pattern-based model

In the following, we refer to a particular skiving stock problem with given input data as an *instance* $E = (m, l, L, b)$ of the SSP with $l = (l_1, \dots, l_m)^\top$ and $b = (b_1, \dots, b_m)^\top$. Any feasible arrangement of items which leads to a final product of minimum length L is called *(packing) pattern* of E. We always represent a pattern by a nonnegative vector $a = (a_1, \dots, a_m)^\top \in \mathbb{Z}_+^m$. There, $a_i \in \mathbb{Z}_+$ denotes the number of items of type $i \in I$ that is contained in the considered pattern. Note that $a \in \mathbb{Z}_+^m$ only provides information as to the total number, but not the specific positioning or order of the necessary items. For a given instance E the set of all patterns is defined by $P_E := \left\{a \in \mathbb{Z}_+^m \mid l^\top a \geq L\right\}$. Let $\widetilde{J} := \widetilde{J}(E)$ be an index set of P_E and $x_j \in \mathbb{Z}_+$ the number, how often the pattern $a^j = \left(a_{1j}, \dots, a_{mj}\right)^\top \in P_E$ is used in the optimization. Then we obtain, (cf. Zak, 2003),

Zak's model of the SSP

$$
\begin{aligned}
& z = \sum_{j \in \widetilde{J}} x_j \to \max \\
\text{s.t.} \quad & \sum_{j \in \widetilde{J}} a_{ij} x_j \leq b_i, && i \in I, \\
& x_j \in \mathbb{Z}_+, && j \in \widetilde{J}.
\end{aligned}
$$

Note that this model has an infinite number of variables, but in every feasible solution there are only a finite number of nonzero variables. In contrast, the feasible set of the continuous relaxation contains (feasible) solutions with an infinite number of positive x-variables. Therefore, we would like to present a method to obtain a finite formulation of Zak's model. To this end, only *minimal patterns* of E are considered. A pattern $a \in P_E$ is called minimal if there does not exist any pattern $\tilde{a} \in P_E$ such that $\tilde{a} \neq a$ and $\tilde{a} \leq a$ hold (componentwise). Let $P_E^\star$ denote the set of minimal patterns and $J := J(E)$ a corresponding index set. Since $l^\top a < 2L$ is a necessary condition for $a \in P_E^\star$ we obtain $\left|P_E^\star\right| = |J| < \infty$. Now, we can formulate the

Standard model of the SSP

$$
\begin{aligned}
& z^S = \sum_{j \in J} x_j \to \max \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \leq b_i, && i \in I, && (1) \\
& x_j \in \mathbb{Z}_+, && j \in J. && (2)
\end{aligned}
$$

The objective function maximizes the total number of connected final products, whereas constraints (1) observe the item supply limitations. In most cases, it is not reasonable or even not possible to have all patterns available prior to the optimization process due to the huge cardinality of $P_E^\star$. Hence, common solvers, like CPLEX, usually cannot be applied to tackle this problem. However, at least the continuous relaxation of the standard model can be solved efficiently by *column generation*. In order to solve the ILP, branch-and-price techniques can be applied as in the context of one-dimensional cutting, see Belov and Scheithauer (2006) for instance. Note that, in this case, the computational behavior strongly depends on the choice of an appropriate branching rule. A further way to tackle the integer problem consists in the consideration of other modeling approaches.

3. A position-indexed and an order-indexed model

In this section, we assume the item lengths to be of strictly decreasing order, i.e.,

$$L > l_1 > l_2 > \cdots > l_m > 0.$$

If such an order is not given from the beginning, we can easily obtain it by a sorting algorithm, e.g. merge sort, in $\mathcal{O}(m \cdot \log m)$ operations.

3.1. The arcflow model

For a given instance E we define the maximum length of a minimal pattern

$$v_{max} := \max\left\{l^\top a \mid a \in P_E^\star\right\}.$$

Then, we consider the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ given by the set of vertices $\mathcal{V} = \{0, 1, \dots, v_{max}\}$ and the set of arcs

$$\mathcal{E} := \left\{(p, q) \in \mathcal{V} \times \mathcal{V} \mid p < L,\ q - p \in \{l_1, \dots, l_m\}\right\}.$$

Note that an arc $(p, q) \in \mathcal{E}$ represents the positioning of an item of length $q - p = l_j \in \{l_1, \dots, l_m\}$ with its left point at vertex $p \in \mathcal{V}$. Each path $s = (v_0, v_1, \dots, v_k)$ in $\mathcal{G}$ from $v_0 = 0$ to $v_k \geq L$ corresponds to a (not necessarily minimal) pattern $a \in P_E$. Since, additionally, our current graph $\mathcal{G}$ contains, in most cases, an oversized number of arcs and vertices, we would like to apply two reduction principles in order to solve these problems:

1. Obviously, we only need to consider vertices that are integer linear combinations of the given item lengths, since all other nodes cannot lie on a path $s = (v_0, v_1, \dots, v_k)$ from $v_0 = 0$ to $v_k \geq L$. Thus, we can replace our set of vertices with a reduced one, i.e., we define
$$\mathcal{V}' := \left\{v \in \mathcal{V} \mid \exists\, a \in \mathbb{Z}_+^m : l^\top a = v\right\}.$$
2. The first restriction is not sufficient for vertices $v \in \mathcal{V}$ with $v \geq L$ since these nodes can belong to non-minimal patterns. Considering the instance $E = (2, (7, 2), 10, (7, 7))$, we note that $12 \in \mathcal{V}'$, but there is no minimal pattern $a \in P_E^\star$ with $l^\top a = 12$. Let $\mathcal{L} := \left\{l^\top a \mid a \in P_E^\star\right\}$ be the set of the total lengths of the minimum patterns. Then we can replace $\mathcal{V}'$ with $\mathcal{V}'' := \mathcal{V}' \cap (\{0, 1, \dots, L - 1\} \cup \mathcal{L})$.

These reductions also lead us to a reduced set of arcs

$$\mathcal{E}' := \left\{(p, q) \in \mathcal{V}'' \times \mathcal{V}'' \mid p < L,\ q - p \in \{l_1, \dots, l_m\}\right\}.$$

If we want to assign a path s in $\mathcal{G}' = \left(\mathcal{V}'', \mathcal{E}'\right)$ to a minimal pattern $a \in P_E^\star$, this identification is not unique since a does not contain any information about the particular arrangement of the items. We therefore like to restrict our investigations to *monotonically decreasing paths*, i.e., paths whose corresponding item lengths are sorted in descending order. This approach is based on a similar method (cf. Scheithauer, 2008) for the well known arcflow model of the cutting