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Decision Support

Portfolio optimization with disutility-based risk measure

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A R T I C L E I N F O

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A B S T R A C T

In this paper we propose a quantile-based risk measure which is defined using the modified loss distribution according to the decision maker's risk and loss aversion. The properties related to different classes of disutility functions are established. A portfolio selection model in the Mean-Risk framework is proposed and equivalent formulations of the model generating the same efficient frontier are given. The advantages of this approach are investigated using real world data from NYSE. The differences between the efficient frontier of the proposed model and the classical Mean-Variance and Mean-CVaR are quantified and interpreted. Extensive experiments show that the efficient portfolios obtained by using the proposed model exhibit lower risk levels and an increased satisfaction compared to the other two Mean-Risk models.

1. Introduction

Solving the portfolio selection problem relies on models for preference between random variables representing portfolio returns. Choosing a specific model is itself a problem because each type of models assumes a different vision on choice under risk, different theoretical basis with strengths and weaknesses, and different degrees of computational tractability. In this paper, we present a new *Mean-Risk* model of portfolio selection. Its novelty consists in using of a new risk measure based on the modified loss distribution function according to the decision maker's risk and loss aversion preferences. These preferences are usually described by increasing, smooth and concave utility functions, but studies in behavioral finance showed that people systematically violate expected utility theory, see for example Camerer, Kagel, and Roth (1995). Kahneman and Tversky (1979) and Tversky and Kahneman (1992) have found that decisions are driven especially by loss aversion and the prospect of ending up with less than the initial wealth. There is also the power utility function with loss aversion implemented by Maringer, Kontoghiorghes, Rustem, and Winker (2008) where the returns are adjusted before they are evaluated in the utility function: returns lower than a prescribed threshold are given disproportionate weight. Therefore, the utility changes abruptly resulting in a kinked utility function, see Cremers, Kritzman, and Page (2005); Hagstromer and Binner (2009); Maringer et al. (2008); Sharpe (2007). There is a variety of reasons why decision makers have critical thresholds: some investors might violate loan covenants if their assets fall below a specified value, others face regulatory mandates which require a minimum level of reserves, also in the practice of risk when asset levels fall under the loss threshold fund managers are penalized.

Apart these approaches relying on utility functions, there are the *Mean-Risk* models. Variance was the first risk measure used in portfolio optimization. The *Mean-Variance* (*MV*) methodology proposed by Markowitz (1952) has played a crucial role in portfolio theory and provided the fundamental basis for the development of a large part of the modern financial theory applied to the portfolio optimization problem; moreover, in his recent paper (Markowitz, 2012), a thoroughly research on mean–variance approximations to expected utility can be found. But regulations for finance businesses formulate some of the risk management requirements in terms of quantiles of loss distributions. The most commonly used is the *Value at Risk* (*VaR*). *VaR* can be efficiently estimated and managed when underlying risk factors are normally distributed. However, for non-normal distributions, *VaR* may have undesirable properties (see Artzner, Delbaen, Eber, & Heath, 1999) such as the lack of sub-additivity. Also, *VaR* is difficult to control/optimize for discrete distributions, when it is calculated using scenarios. In this case, *VaR* is non-convex and non-smooth as a function of positions, and has multiple local extrema. To alleviate these problems, (Artzner et al., 1999) proposed the *Conditional VaR* (*CVaR*) which is sub-additive and, consequently, coherent. *CVaR* continues to be intensively studied and applied in different contexts. For example, in the context of enhanced indexation, the paper of Roman, Mitra, and Zviarovich (2013) provides a unified framework incorporating *CVaR* and second order stochastic dominance. More detailed discussions on *CVaR* and new advances on its

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estimation and asymptotics can be found in Rockafellar and Uryasev (2002) and Chun, Shapiro, and Uryasev (2012), just to name a few.

The contribution of this paper is threefold. Firstly, we propose a risk measure defined using the modified loss distribution according to the decision maker's risk and loss aversion preferences and establish its properties (Section 2). Our motivation is based on the well known fact well acknowledged in the literature that different categories of investors have different modeling needs which are not well met by standard approaches, see (Spronk & Hallerbach, 1997; Cillo & Delquié, 2014; Wächter & Mazzoni, 2013). Investors preferences are often, but not exclusively related to utility functions. Many other different ideas of modeling them were proposed in the literature. For example, investor's preferences can be captured through various types of constraints such as cardinality constraints in portfolio optimization as in Chang, Meade, Beasley, and Sharaiha (2000) and Woodside-Oriakhi, Lucas, and Beasley (2011), or in energy planning as in Chang (2014), crop planning as in Rădulescu and Rădulescu (2012) and Rădulescu and Rădulescu (2014), socially responsible investments as in Hallerbach, Ning, Soppe, and Spronk (2004), just to name a few. Also, to accommodate the preferences of non-standard investors (expressed as additional stochastic and deterministic objectives such as liquidity, dividends, number of securities in a portfolio, social responsibility,...), portfolio selection models with multiple criteria that ensures portfolio suitability are developed in Steuer, Qi, and Hirschberger (2007). Moreover, the very choice of the risk measure used in the portfolio model is a manifestation of preferences. Apart the classical risk measures, new ones were designed to cover various risk profiles, see for example the Conditional Average (inspired by *CVaR* but using two confidence levels) introduced by Krzemienowski (2009). The idea of using more than one confidence level is also present in Kalinchenko, Uryasev, and Rockafellar (2012) where risk preferences are calibrated by the coefficients in the mixed *CVaR* deviation. But usually investors preferences are related to utility functions. Up to now, in conjunction with a *Mean-Risk* model (most frequently with *Mean-Variance*), investor's utility function was used to determine the preferred portfolio out of the investment set represented by the *Mean-Risk* efficient frontier, see (De Giorgi & Hens, 2009; Hens & Mayer, 2014; Kroll, Levy, & Markowitz, 1984; Levy & Levy, 2004; Pirvu & Schulze, 2012). In this paper, instead of taking into account investor's preferences only in this second phase, we bring them into the risk measure by using her/his utility function that captures the investor's risk and loss aversion; consequently, preferences are fully taken into consideration right from the first phase in which the efficient frontier is determined. Thus, this paper addresses one of the main universally acknowledged streams of concerns: the challenge to balance the properties of the risk measure with behavioral and practical considerations. Secondly, we consider a bi-objective portfolio selection model in the *Mean-Risk* framework using the risk measure previously defined. and give different equivalent representation of the efficient frontier (Section 3). We give three equivalent formulations of the model. They are equivalent in the sense that they produce the same efficient frontier. The equivalence between the three models is proven for the *Mean-Variance* efficient frontier in Steinbach (2001), for the *Mean-Regret* efficient frontier in Dembo and Rosen (1999) and for the *Reward-CVaR* efficient frontier in Krokhmal, Palmquist, and Uryasev (2002). Thirdly, we investigate the practical performances of the model using real data from New York Stock Exchange (Section 4). We present the forecast procedure used and the out-of-sample analysis assessing the reliability of the forecast (Section 4.1). We compare the efficient frontier of the proposed model with the *Mean-CVaR*and *Mean-Variance* frontiers quantifying the differences and similarities between them (Section 4.2). Moreover, we assess the investor's benefits of using the proposed model: out-of-sample experiments show that the efficient portfolios obtained by using it exhibit lower risk levels (at the same return levels).

2. Disutility-based risk measures

Let n be the number of securities available for the portfolio. The key random inputs in the portfolio management problem are the asset prices at the end of the planning horizon denoted by $p(\omega) = (p_1(\omega), \dots, p_n(\omega)),\ \omega \in \Omega$ or simply by $\boldsymbol{p}$ (we use bold symbols for vectors). The set Ω represents the set of future states of knowledge and has the mathematical structure of a probability space with a probability measure P for comparing the likelihood of future states ω. Let $l(\boldsymbol{x}, \boldsymbol{p})$ be the loss associated with the decision vector $\boldsymbol{x} \in X \subset \mathbb{R}^n$ and the random vector $\boldsymbol{p}$, where $\boldsymbol{x}$ is interpreted as a portfolio and X is the set of available portfolios subject to various constraints. The loss equals to the difference between the initial wealth W_0 and the final random wealth, $l(\boldsymbol{x}, \boldsymbol{p}) = W_0 - W$, where $W = \boldsymbol{x}^T \boldsymbol{p}$. Positive outcomes of loss function are disliked, while negative outcomes are welcome because they represent gains. For each $\boldsymbol{x} \in X$, the loss $l(\boldsymbol{x}, \boldsymbol{p})$ is a random variable having a distribution in $\mathbb{R}$ induced by that of $\boldsymbol{p}$. Throughout this paper, the loss function can have a more general form if it is continuous in $\boldsymbol{x}$, measurable in $\boldsymbol{p}$ and $E(|l(\boldsymbol{x}, \boldsymbol{p})|) < \infty\ \forall \boldsymbol{x} \in X$. The underlying probability distribution of $\boldsymbol{p}$ in $\mathbb{R}^n$ is assumed to have the probability density function (pdf) $g(\boldsymbol{p})$, $\boldsymbol{p} \in \mathbb{R}^n$.

Given z a level of losses, the cumulative distribution function (cdf) of $l(\boldsymbol{x}, \boldsymbol{p})$ is defined by $G_{l(\boldsymbol{x},\boldsymbol{p})}(z) = P(\{\boldsymbol{p} \,|\, l(\boldsymbol{x}, \boldsymbol{p}) \le z\}) = \int_{l(\boldsymbol{x},\boldsymbol{p}) \le z} g(\boldsymbol{p}) d\boldsymbol{p}$ and is assumed continuous with respect to z. Let $G^{\leftarrow}_{l(\boldsymbol{x},\boldsymbol{p})} : (0, 1) \to \mathbb{R}$ be the $\alpha-$quantile function, given by $G^{\leftarrow}_{l(\boldsymbol{x},\boldsymbol{p})}(\alpha) = \min_{G_{l(\boldsymbol{x},\boldsymbol{p})}(z) \ge \alpha} z$. Within risk management, it is called the *Value at Risk* of the loss $l(\boldsymbol{x}, \boldsymbol{p})$ at a probability level of $\alpha \in (0, 1)$, denoted by $VaR_\alpha(l(\boldsymbol{x}, \boldsymbol{p}))$ or $z_\alpha(\boldsymbol{x})$. Artzner et al. (1999) introduced the concept of coherent risk measure. The *Conditional Value at Risk* ($CVaR_\alpha$)of the loss $l(\boldsymbol{x}, \boldsymbol{p})$ at probability level $\alpha \in (0, 1)$ proved to be coherent, see for example Pflug and Uryasev (2000). The dedicated notation which associates any portfolio $\boldsymbol{x} \in X$ to its corresponding $CVaR_\alpha$ is $\phi_\alpha : X \to \mathbb{R}$ given by

$$\phi_\alpha(\boldsymbol{x}) = \frac{1}{1-\alpha} \int_{l(\boldsymbol{x},\boldsymbol{p}) \ge z_\alpha(\boldsymbol{x})} l(\boldsymbol{x}, \boldsymbol{p}) g(\boldsymbol{p}) d\boldsymbol{p}. \tag{1}$$

Generally, in the literature, $l(\boldsymbol{x}, \boldsymbol{p}) = W_0 - W$ and consequently, for a given probability distribution of $\boldsymbol{p}$, $CVaR_\alpha$ will be the same for all investors whatever their particular profiles of loss aversion. But in real life, an investor has a certain risk profile and also a critical loss level; it is the case of an investor whose lifestyle might change drastically if a certain critical loss level is reached. Therefore, a more realistic approach is to consider that the decision maker is characterized by an increasing convex disutility function D with loss aversion which exhibits a kink at this critical loss level θ. When θ is reached, the perception of losses changes abruptly: the losses higher than this critical threshold are given disproportionate weight in accordance with a loss aversion parameter λ which yields a kink on the disutility function located at the critical loss value. An example of such a disutility function D that captures the investor's risk and loss aversion can be written based on the utility function U defined in Adler and Kritzman (2007); Cremers et al. (2005):

$$U(z) = \begin{cases} \ln(1+z), & \text{for } z \ge \theta \\ \lambda(z-\theta) + \ln(1+\theta), & \text{for } z < \theta, \end{cases} \tag{2}$$

where $D(z) = -U(-z),\ \forall z \in \mathbb{R}$. This function is also similar to the piecewise linear loss-averse utility used in Fortin and Hlouskova (2011). More examples of disutility functions with loss aversion, can be found in Maringer et al. (2008) and for the effect of utility functions on the optimal portfolio see also Yu, Pang, Troutt, and Hou (2009) and Çanakoğlu and Özekici (2010).

When making decisions by considering the disutility of the loss as previously described, we define the *Conditional Value at Risk of the*