European Journal of Operational Research 251 (2016) 575–585

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

A cardinal dissensus measure based on the Mahalanobis distance

T. González-Arteaga [a,*], J. C. R. Alcantud [b], R. de Andrés Calle [c]

[a] Presad Research Group, Facultad de Ciencias, Universidad de Valladolid, Valladolid E47011, Spain
[b] Borda Research Unit and Multidisciplinary Institute of Enterprise (IME), Universidad de Salamanca, Salamanca E37007, Spain
[c] Borda Research Unit, Presad Research Group and IME, Universidad de Salamanca, Salamanca E37007, Spain

ARTICLE INFO

Article history:
Received 20 March 2015
Accepted 13 November 2015
Available online 2 December 2015

Keywords:
Decision analysis
Consensus/dissensus
Cardinal profile
Mahalanobis distance
Correlation

ABSTRACT

In this paper we address the problem of measuring the degree of consensus/dissensus in a context where experts or agents express their opinions on alternatives or issues by means of cardinal evaluations. To this end we propose a new class of distance-based consensus model, the family of the Mahalanobis dissensus measures for profiles of cardinal values. We set forth some meaningful properties of the Mahalanobis dissensus measures. Finally, an application over a real empirical example is presented and discussed.

1. Introduction

In Decision Making Theory and its applications, consensus measurement and its reaching in a society (i.e., a group of agents or experts) are relevant research issues. Many studies investigating the aforementioned subjects have been carried out under several frameworks (see Cabrerizo, Moreno, Pérez, & Herrera-Viedma, 2010; Dong, Xu, & Li, 2008; Dong, Xu, Li, & Feng, 2010; Dong & Zhang, 2014; Fedrizzi, Fedrizzi, & Marques Pereira, 2007; Fu & Yang, 2012; Herrera-Viedma, Herrera, & Chiclana, 2002; Liu, Liao, & Yang, 2015; Palomares, Estrella-Liébana, Martínez, & Herrera, 2014; Wu & Chiclana, 2014a, 2014b; Wu, Chiclana, & Herrera-Viedma, 2015 among others) and based on different methodologies (Chiclana, Tapia García, del Moral, & Herrera-Viedma, 2013; Cook, 2006; Eklund, Rusinowska, & de Swart, 2008; Eklund, Rusinowska, & Swart, 2007; Fedrizzi et al., 2007; Fu & Yang, 2010, 2011; Gong, Zhang, Forrest, Li, & Xu, 2015; González-Pachón & Romero, 1999; Liu et al., 2015; Palomares & Martínez, 2014 among others).

Since the seminal contribution by Bosch (2005) several authors have addressed the consensus measurement topic from an axiomatic perspective. Earlier analyses can be mentioned, e.g., Hays (1960) or Day and McMorris (1985). This issue is also seen as the problem of combining a set of ordinal rankings to obtain an indicator of their 'consensus', a term with multiple possible meanings (Martínez-Panero, 2011).

Generally speaking, the usual axiomatic approaches assume that each individual expresses his or her opinions through ordinal preferences over the alternatives. A group of agents is characterized by the set of their preferences – their preference profile. Then a consensus measure is a mapping which assigns to each preference profile a number between 0 and 1. The assumption is made that the higher the values, the more consensus in the profile.

Technical restrictions on the preferences provide various approaches in the literature. In most cases the agents are presumed to linearly order the alternatives (see Bosch, 2005 or Alcalde-Unzu & Vorsatz, 2013). Since this assumption seems rather demanding (especially as the number of alternatives grows), an obvious extension is to allow for ties. This is the case where the agents have complete preorders on the alternatives (e.g., García-Lapresta & Pérez-Román, 2011). Alcantud, de Andrés Calle, and Cascón (2013a, 2015) take a different position. They study the case where agents have dichotomous opinions on the alternatives, a model that does not necessarily require pairwise comparisons.

Notwithstanding the use of different ordinal preference frameworks, the problem of how to measure consensus is an open-ended question in several research areas. This fact is due to that methodology used in each case is a relevant element in the problem addressed. To date various methods have been developed to measure consensus under ordinal preference structures based on distances and association measures like Kemeny's distance, Kendall's coefficient, Goodman-Kruskal's index and Spearman's coefficient among others (see e.g., Cook & Seiford, 1982; Goodman & Kruskal, 1979; Kemeny, 1959; Kendall & Gibbons, 1990; Spearman, 1904).

In this paper we first tackle the analysis of coherence that derives from profiles of cardinal rather than ordinal evaluations. Modern

* Corresponding author. Tel.: +34 983185871.
E-mail addresses: teresag@eio.uva.es (T. González-Arteaga), jcr@usal.es (J.C.R. Alcantud), rocioac@usal.es (R. de Andrés Calle).

http://dx.doi.org/10.1016/j.ejor.2015.11.019
0377-2217/

convention applies the term cardinal to measurements that assign significance to differences (cf., Basu, 1982; Chiclana, Herrera-Viedma, Alonso, & Herrera, 2009; High & Bloch, 1989). In contrast ordinal preferences only permit to order the alternatives from best to worst without any additional information. To see how this affects the analysis of our problem, let us consider a naive example of a society with two agents. They evaluate two public goods with monetary amounts. One agent gives a value of 1€ for the first good and 2€ for the second good. The other agent values these goods at 10€ and 90€ respectively. If we only use the ordinal information in this case, we should conclude that there is unanimity in the society: all members agree that 'good 2 is more valuable than good 1'. However the agents disagree largely. Therefore, the subtleties of cardinality clearly have an impact when we aim at measuring the cohesiveness of cardinal evaluations.

Unlike previous references, we adopt the notion of dissensus measure as the fundamental concept. This seems only natural because it resembles more the notion of a "measure of statistical dispersion", in the sense that 0 captures the natural notion of unanimity as total lack of variability among agents, and then increasingly higher numbers mean more disparity among evaluations in the profile.[1]

In order to build a particular dissensus measure we adopt a distance-based approach. Firstly, one computes the distances between each pair of individuals. Then all these distances are aggregated. In our present proposal the distances (or similarities) are computed through the Mahalanobis distance (Mahalanobis, 1936). We thus define the class of Mahalanobis dissensus measures.

The Mahalanobis distance plays an important role in Statistics and Data Analysis. It arises as a natural generalization of the Euclidean distance. A Mahalanobis distance accounts for the effects of differences in scales and associations among magnitudes. Consequently, building on the well-known performance of the Mahalanobis distance, our novel proposal seems especially fit for the cases when the measurement units of the issues are different, e.g., performance appraisal processes when employees are evaluated attending to their productivity and their leadership capacity; or where the issues are correlated. For example, evaluation of related public projects. An antecedent for the weaker case of profiles of preferences has been provided elsewhere, cf. Alcantud, de Andrés Calle, and González-Arteaga (2013b), and an application to comparisons of real rankings on universities worldwide is developed. Here we apply our new indicator to a real situation, namely, economic forecasts made by several agencies. Since the forecasts concern economic quantities, they have an intrinsic value which is naturally cardinal and also there are relations among them.

The paper is structured as follows. In Section 2, we introduce basic notation and definitions. In Section 3, we set forth the class of the Mahalanobis dissensus measures and their main properties. Section 4 provides a comparison of several Mahalanobis dissensus measures. Next, a practical application with discussion is given in Section 5. Finally, we present some concluding remarks. Appendices contain proofs of some properties and a short review in matrix algebra.

2. Notation and definitions

This section is devoted to introduce some notation and a new concept in order to compare group cohesiveness: namely, dissensus measures. Then, a comparison with the standard approach is made. We partially borrow notation and definitions from Alcantud et al. (2013b). In addition, we use some elements of matrix analysis that we recall in Appendix B to make the paper self-contained.

Let $X = \{x_1, \ldots, x_k\}$ be the finite set of k issues, options, alternatives, or candidates. It is assumed that X contains at least two options, i.e., the cardinality of X is at least 2. Abusing notation, on occasions we refer to issue x_s as issue s for convenience. A population of agents or experts is a finite subset $\mathbf{N} = \{1, 2, \ldots, N\}$ of natural numbers. To avoid trivialities we assume $N > 1$.

We consider that each expert evaluates each alternative by means of a quantitative value. The quantitative information gathered from the set of N experts on the set of k alternatives is summarized by an $N \times k$ numerical matrix M:

$$M = \left(M_{ij}\right)_{N\times k}$$

We write M_i to denote the evaluation vector of agent i over the issues (i.e., row i of M) and M^j to denote the vector with all the evaluations for issue j (i.e., column j of M). For convenience, $(1)_{N\times k}$ denotes the $N \times k$ matrix whose cells are all equal to 1 and $\mathbf{1}_N$ denotes the column vector whose N elements are equal to 1. We write $\mathbb{M}_{N\times k}$ for the set of all $N \times k$ real-valued matrices. Any $M \in \mathbb{M}_{N\times k}$ is called a *profile*.

Any permutation σ of the experts $\{1, 2, \ldots, N\}$ determines a profile M^σ by permutation of the rows of M: row i of the profile M^σ is row $\sigma(i)$ of the profile M. Similarly, any permutation π of the alternatives $\{1, 2, \ldots, k\}$ determines a profile ${}^\pi M$ by permutation of the columns of M: column i of the profile ${}^\pi M$ is column $\pi(i)$ of the profile M.

For each profile $M \in \mathbb{M}_{N\times k}$, its restriction to *subprofile* on the issues in $I \subseteq X$, denoted M^I, arises from exactly selecting the columns of M that are associated with the respective issues in I (in the same order). And for simplicity, if $I = \{j\}$ then $M^I = M^{\{j\}} = M^j$ is column j of M. Any partition $\{I_1, \ldots, I_s\}$ of $\{1, 2, \ldots, k\}$, that we identify with a partition of X, generates a *decomposition*of M into subprofiles $M^{I_1}, \ldots, M^{I_s}$.[2]

A profile $M \in \mathbb{M}_{N\times k}$ is *unanimous* if the evaluations for all the alternatives are the same across experts. In matrix terms, the columns of $M \in \mathbb{M}_{N\times k}$ are constant, or equivalently, all rows of the profile are coincident.

An *expansion* of a profile $M \in \mathbb{M}_{N\times k}$ of $\mathbf{N}$ on $X = \{x_1, \ldots, x_k\}$ is a profile $\bar{M} \in \mathbb{M}_{\bar{N}\times k}$ of $\bar{\mathbf{N}} = \{1, \ldots, N, N+1, \ldots, \bar{N}\}$ on $X = \{x_1, ..., x_k\}$, such that the restriction of $\bar{M}$ to the first N experts of $\mathbf{N}$ coincides with M.

Finally, a *replication* of a profile $M \in \mathbb{M}_{N\times k}$ of the society $\mathbf{N}$ on $X = \{x_1, \ldots, x_k\}$ is the profile $M \uplus M \in \mathbb{M}_{2N\times k}$ obtained by duplicating each row of M, in the sense that rows t and $N+t$ of $M \uplus M$ are coincident and equal to row t of M, for each $t = 1, \ldots, N$.

We now define a dissensus measure as follows:

Definition 1. A *dissensus measure* on $\mathbb{M}_{N\times k}$ is a mapping defined by $\delta : \mathbb{M}_{N\times k} \to [0, \infty)$ with the property:

(i) *Unanimity*: for each $M \in \mathbb{M}_{N\times k}$, $\delta(M) = 0$ if and only if the profile $M \in \mathbb{M}_{N\times k}$ is unanimous.
We also define a *normal* dissensus measure as a dissensus measure that additionally verifies:
(ii) *Anonymity*: $\delta(M^\sigma) = \delta(M)$ for each permutation σ of the agents and $M \in \mathbb{M}_{N\times k}$.
(iii) *Neutrality*: $\delta({}^\pi M) = \delta(M)$ for each permutation π of the alternatives and $M \in \mathbb{M}_{N\times k}$.

This definition does not attempt to state dissensus by opposition to consensus. The literature usually deals with a formulation of consensus where the higher the index, the more coherence in the society's opinions. The terms consensus and dissensus should not be taken as formal antonyms, especially because a universally accepted definition of consensus is not available and we do not intend to give an absolute concept of dissensus. However, consensus measures in the sense of Bosch (see Bosch, 2005, Definition 3.1) verify anonymity and neutrality (see also Alcantud et al., 2013b, Definition 1), and from

[1] As a remote antecedent of this position, we note that statistically variance-based methods are commonly employed to measure consensus of verbal opinions (cf., Hoffman, 1994, and Mejias, Shepherd, Vogel, & Lazaneo, 1996.)

[2] A partition of a set S is a collection of pairwise disjoints non-empty subsets of S whose union is S.