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Decision Support

The cost of getting local monotonicity

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ABSTRACT

Committees with yes-no-decisions are commonly modeled as simple games and the ability of a member to influence the group decision is measured by so-called power indices. For a weighted game we say that a power index satisfies local monotonicity if a player who controls a large share of the total weight vote does not have less power than a player with a smaller voting weight.

In (Holler, 1982) Manfred Holler introduced the Public Good index. In its unnormalized version, i.e., the raw measure, it counts the number of times that a player belongs to a minimal winning coalition. Unlike the Banzhaf index, it does not count the remaining winning coalitions in which the player is crucial. Holler noticed that his index does not satisfy local monotonicity, a fact that can be seen either as a major drawback (Felsenthal & Machover, 1998, 221 ff.)or as an advantage (Holler & Napel 2004).

In this paper we consider a convex combination of the two indices and require the validity of local monotonicity. We prove that the cost of obtaining it is high, i.e., the achievable new indices satisfying local monotonicity are closer to the Banzhaf index than to the Public Good index. All these achievable new indices are more solidary than the Banzhaf index, which makes them as very suitable candidates to divide a public good.

As a generalization we consider convex combinations of either: the Shift index, the Public Good index, and the Banzhaf index, or alternatively: the Shift Deegan–Packel, Deegan–Packel, and Johnston indices.

1. Introduction

Consider a set of players who jointly make decisions under a given set of rules. Here we specialize to simple games and subclasses thereof. Power indices address the question of how much power collective decision rules, like a weighted (voting) rule, award to each individual player: is player i more or less powerful than player j, and by how much? For an example of an applied voting power analysis in the EU, we refer the interested reader to e.g. Algaba, Bilbao, and Fernández (2007); Bilbao, Fernández, Jiménez, and López (2002); Widgrén (1994). Different power indices measure different aspects of power and there is still a lot of research in order to answer the question which index to choose, see, e.g., Holler and Nurmi (2013). For a recent overview of different power indices see, e.g., Bertini, Freixas, Gambarelli, and Stach (2013). Many of these indices are based on decisiveness. A player is called critical in a coalition if his/her deletion in the coalition changes its status from winning to losing, so that the individual is decisive or crucial for it. All power indices, the classical and the newly introduced ones, considered in this paper are indeed based on counting different types of decisiveness for players in coalitions.

Some particular decision rules arise from so-called weighted games. Here each player $i \in \{1, \dots, n\}$ has a specific voting weight w_i and a collective decision requires enough supporters such that their total weight equals or surpasses a decision quota q. Let p_i be the power value assigned to player i by a power index. The power index is called *locally monotonic* if, for each pair of players i and j, $w_i \geq w_j$ implies $p_i \geq p_j$, i.e., a player i who controls a large share of vote does not have less power than a player j with smaller voting weight. Local monotonicity is considered as an essential requirement for power measures by many authors. Felsenthal and Machover (1998, 221 ff.), for instance, argue that any a priori measure of power that violates local monotonicity, LM for brevity, is 'pathological' and should be disqualified as serving as a valid yardstick for measuring power. On the other hand, e.g. in Holler and Napel (2004), it is argued that local non-monotonicity is a very valuable property of a power index, since it can reveal certain properties of the underlying decision rule that are overlooked otherwise.

Local monotonicity is an implication of the *dominance* postulate which is based on the *desirability* relation as proposed by Isbell (1958). This property formalizes that a player i is at least as desirable as a player j if for any coalition S, such that j is not in S and the

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union of S and $\{j\}$ is a winning coalition, i.e., is able to pass the collective decision at hand, the union of S and $\{i\}$ is also a winning coalition. A power index satisfies *dominance* if $p_i \ge p_j$ whenever i dominates j, i.e., when player i is at least as desirable as player j.

Freixas and Gambarelli (1997) use desirability to define reasonable power measures and note that the dominance postulate implies local monotonicity. In this paper we will consider the Public Good, the Banzhaf, the Shift, the Shift Deegan–Packel, the Deegan–Packel, the Johnston index and convex combinations thereof. Since the Deegan–Packel index (Deegan & Packel, 1978), and the Public Good index (see Holler, 1982; Holler and Packel, 1983) violate local monotonicity, they also violate the dominance postulate. Moreover, any violation of local monotonicity for the Deegan–Packel index implies a violation of the Shift Deegan–Packel index (see Alonso-Meijide, Freixas, & Molinero, 2012) and any violation of the local monotonicity for the Public Good index implies a violation of the Shift index (see Alonso-Meijide & Freixas, 2010). It is well-known that the Banzhaf (1965) and Johnston (1978) indices satisfy the dominance postulate and therefore local monotonicity. If one or several power indices violate LM then a convex combination with another power index, that does not violate LM, yields a power index that also does not violate LM as long as the weight of the latter index is large enough. To study how large this has to be is the purpose of this paper.

Some works are devoted to verify the properties of dominance or local monotonicity (among others) for some power indices and to show their absence for some other power indices (see among others, Felsenthal and Machover (1995) or Freixas, Marciniak, and Pons (2012)). Other works are devoted to study subclasses of games for which a given power index not fulfilling local monotonicity satisfies it for such a subclass of games (see for instance, Holler, Ono, and Steffen (2001) and Holler and Napel (2004) for the Public Good index). Here we will also make a new contribution of this type, i.e., we consider two new subclasses of games for which the Public Good index satisfies local monotonicity.

Proportionality of power and weights can be seen as a generalization of local monotonicity, i.e., proportionality implies local monotonicity. For the classical power indices this property is satisfied for a subset of weighted games only. Power indices which generally satisfy this property are constructed in Kaniovski and Kurz (2015).

In this paper we modify the Public Good index with the purpose to achieve a set of new power indices being local monotonic and more solidary than the Banzhaf index. These two properties make those achievable power indices (if they exist) well-situated as yardstick for doing a fair division of a public good. The idea of such modification is nothing else than an hybrid between the original Public Good index and the Banzhaf index. It will turn out that the cost of obtaining local monotonicity is rather high, i.e., the achievable new indices satisfying local monotonicity are closer to the Banzhaf index than to the Public Good index. However, these indices stress more in minimal winning coalitions, as the Public Good index does, than in the rest of crucial winning coalitions, with goes in the direction of Riker's size principle (Riker, 1962). The final result allows to find new indices being locally monotonic and being more solidary than the Banzhaf index, which makes them as good alternatives for the fair division of a public good among participants in the voting procedure.

The idea developed previously naturally extends when the raw Shift index is incorporated to the duo formed by the raw Public Good and raw Banzhaf indices. Local monotonic indices which are convex combinations of the three given raw indices are a further target of our research.

As an extension we do a similar study for convex combinations of the raw Johnston index, the raw Deegan–Packel index, and the raw Shift Deegan–Packel index.

The paper is organized as follows: In Section 2 we introduce the basic notation of games and power indices. Two subclasses of weighted games satisfying local monotonicity are presented in Section 3. The concept of considering convex combinations of some power indices as a new power index is outlined in Section 4. The cost of local monotonicity is introduced in the same section. Additionally we prove some structural results. An integer linear programming approach to compute the cost of local monotonicity is presented in Section 5. With the aid of the underlying algorithm we are able to state some exact values and lower bounds for the cost of local monotonicity in Section 6. The set of all convex multipliers leading to a locally monotonic power index is the topic of Section 7. We end with a conclusion in Section 8.

2. Notation, games and indices

In the following we will denote the set of players, which jointly make a decision, by N and assume w.l.o.g. that the players are numbered from 1 to n, i.e., $N = \{1, \ldots, n\}$. Here we restrict ourselves to binary decisions, i.e., each player can either vote 1, meaning 'yes', or 0, meaning 'no', on a certain issue. For the readers convenience we collect all necessary definitions briefly at this place. For a more extensive introduction we refer to Felsenthal and Machover (1998), Taylor and Zwicker (1999).

We call a subset $S \subseteq N$, collecting the 'yes'-voters, *coalition*. A (binary) decision rule is formalized as a mapping $v : 2^N \to \{0, 1\}$ from the set of possible coalitions to the set of possible aggregated decisions. It is quite natural to require that the aggregated decision transfers the players decision if they all coincide and that an enlarged set of supporters should not turn the decision from 'yes' to 'no':

Definition 1. A *simple game* is a mapping $v : 2^N \to \{0, 1\}$ such that $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \le v(T)$ for all $S \subseteq T \subseteq N$.

Having local monotonicity in mind we additionally require that the players are linearly ordered according to their capabilities to influence the final group decision. This can be formalized, as already indicated in the introduction, with the desirability relation introduced in Isbell (1958). Intuitively, the dominance (or desirability) relation is an attempt to formalize the intuitive notion that underlies under the expression:'i has at least as power than j', while the equivalence between i and j formalizes the expression 'i and j have the same power'.

Definition 2. We write $i \sqsupset j$ (or $j \sqsubset i$) for two players $i, j \in N$ of a simple game v if we have $v\Big(\{i\} \cup S \backslash \{j\}\Big) \ge v(S)$ for all $\{j\} \subseteq S \subseteq N \backslash \{i\}$ and we abbreviate $i \sqsupset j, j \sqsupset i$ by $i \square j$.

In words we say that i dominates j for $i \sqsupset j$ and we say that i and j are equivalent for $i \square j$.

Definition 3. A simple game v is called *complete* if the binary relation $\sqsupset$ is a total preorder, i.e.,

(1) $i \sqsupset i$ for all $i \in N$,
(2) $i \sqsupset j$ or $j \sqsupset i$ for all $i, j \in N$, and
(3) $i \sqsupset j, j \sqsupset h$ implies $i \sqsupset h$ for all $i, j, h \in N$.

We call a coalition S of a simple game v *winning* if $v(S) = 1$ and *losing* otherwise. Each simple game is uniquely characterized by its set $\mathcal{W}$ of winning coalitions (or its set $\mathcal{L}$ of losing coalitions). A winning coalition S such that each of its proper subsets is losing is called a *minimal winning* coalition. The set $\mathcal{M}$ of minimal winning coalitions is already sufficient to uniquely characterize a simple game. For complete games the defining set of winning coalitions can be further reduced. A minimal winning coalition S is called *shift-minimal* if for each pair of players i, j with $i \in S, j \notin S, i \sqsupset j, j \not\sqsupset i$ we have $v(S \backslash \{i\} \cup \{j\}) = 0$, i.e., replacing a player by a (properly) dominated player turns the coalition into a losing one. With this, each complete game is uniquely characterized by its set $\mathcal{S}$ of shift-minimal winning coalitions.

A very transparent form of dominance is induced by weights.