European Journal of Operational Research 250 (2016) 754–759

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Discrete Optimization

A new lower bound for online strip packing☆

Guosong Yu[a,*], Yanling Mao[b], Jiaoliao Xiao[b]

[a] Department of Mathematics, Nanchang University, Nanchang 330031, PR China
[b] Department of Management Science and Engineering, Nanchang University, Nanchang 330031, PR China

ARTICLE INFO

Article history:
Received 12 January 2015
Accepted 6 October 2015
Available online 22 October 2015

Keywords:
Packing
Strip packing
Online algorithm
Competitive ratio

ABSTRACT

In this paper, we consider the online strip packing problem, in which a list of online rectangles has to be packed without overlap or rotation into a strip of width 1 and infinite length so as to minimize the required height of the packing. We derive a new improved lower bound of $(3+\sqrt{5})/2 \approx 2.618$ for the competitive ratio for this problem. This result improves the best known lower bound of 2.589.

1. Introduction

In this paper, we consider online strip packing of rectangles. Rectangles arrive one by one in an online fashion and have to be packed into a strip of width 1 and infinite length without known any information about future rectangles. The rectangles must be packed without overlap and rotation and couldn't be moved when they are already packed. The objective is to minimize the total height of the packing. The strip packing problem was first considered by Baker, Coffman, and Rivest (1980). They showed that this problem is $NP-$Hard. Strip packing has many real-world applications in manufacturing, logistics, and computer science, e.g., VLSI layout design, stock cutting problem.

To evaluate the performance of an online algorithm we adopt competitive analysis. For any list of rectangles L, the height of a strip used by algorithm A and by the optimal solution is denoted by $A(L)$ and $OPT(L)$, respectively. The (absolute) competitive ratio of A, denoted by R_A, is given by

$$R_A = \sup_L \left\{ \frac{A(L)}{OPT(L)} \right\}.$$

The asymptotic competitive ratio R_A^∞ of A is defined by

$$R_A^\infty = \limsup_{n\to\infty} \sup_L \left\{ \frac{A(L)}{OPT(L)} \middle| OPT(L) = n \right\}.$$

For the offline strip packing problem, Coffman, Garey, Johnson, and Tarjan (1980) presented the algorithms NFDH and FFDH with asymptotic competitive ratios of 2 and 1.7, respectively. An AFPTAS was given by Kenyon and Rémila (2000). Sleator (1980) presented an approximation algorithm with an absolute competitive ratio of 2.5. This was independently improved by Schiermeyer (1994) and Steinberg (1997) with algorithms of absolute competitive ratio 2. Harren and van Stee (2009) first broke the barrier of 2 and presented an algorithm with a absolute competitive ratio of 1.936. Then this bound was improved to $5/3+\varepsilon$ for any $\varepsilon > 0$ by Harren, Jansen, Prädel, and van Stee (2014).

For the online strip packing problem, Baker and Schwarz (1983) showed the first fit shelf algorithm has absolute competitive ratio of 6.99. The upper bound was improved to $7/2+\sqrt{10} \approx 6.6623$ by Ye, Han, and Zhang (2009) and Hurink and Paulus (2007) independently. Regarding the lower bound on the competitive ratio for online strip packing, Brown, Baker, and Katseff (1982) derived a lower bound $\rho \geq 2$ on the competitive ratio of any online algorithm by constructing adversary sequences (BBK sequences). Then Johannes (2006) and Hurink and Paulus (2008) improved the bound to 2.25 and 2.43 by studying BBK sequences. Kern and Paulus (2013) finally showed that the BBK sequences can be packed by providing matching upper and lower bounds of $3/2+\sqrt{33}/6 \approx 2.457$. The current best known result $\rho \geq 2.589$ was presented by Harren and Kern (2015) by constructing modified BBK sequences.

A related problem is the multiple-strip packing problem. Zhuk (2006) first considered this problem and showed that there is no approximation algorithm with absolute competitive ratio better than 2 unless $P = NP$ even if there are only two strips. Latter Ye, Han, and Zhang (2011) presented a nearly optimal algorithm with an absolute competitive ratio of $2+\varepsilon$ for any $\varepsilon > 0$. Then

☆ This work was supported by the project of National Natural Science Foundation of China (no. 71263038) and MOE project of Humanities and Social Sciences Foundation (no. 12YJA630091).

* Corresponding author. Tel.: +86 13576089185.
E-mail address: yuguosong@ncu.edu.cn (G. Yu).

http://dx.doi.org/10.1016/j.ejor.2015.10.012
0377-2217/

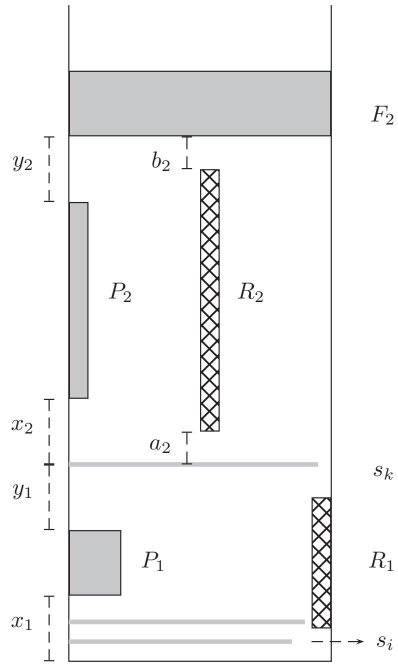

Fig. 1. The list of L_n.

Bougeret, Dutot, Jansen, Otte, and Trystram (2010) presented an approximation algorithm with absolute competitive ratio of 2, which is the best possible unless $P = NP$. For the online multiple-strip packing problem, Ye et al. (2011) designed both randomized and deterministic online algorithms with competitive ratios better than the previous bound 10 presented by Zhuk (2006).

2. The instance construction

In this section, we first describe the construction of new modified BBK sequences $L_n = (J_1, \dots, J_i, \dots)$ in order to get a new lower bound for the competitive ratio of online strip packing, where each item J_i denotes a rectangle of height J_i and width $w(J_i)$. In the end of this section, we will give the reason why we design this type of our sequences.

Assume that there exists a $\rho-$competitive online algorithm A with $\rho < (3+\sqrt{5})/2$. Since Brown et al. (1982) derived a lower bound 2, we suppose that $\rho > 2$ in the following. We can define L_n as the list of items

$(P_1, s_1, \dots, s_k, R_1, P_2, F_2, R_2, \dots, P_{n-1}, F_{n-1}, R_{n-1}, P_n)$.

The sequences L_n consist of four types of rectangles:P_i, s_i, R_i, F_i. For any i, P_i and R_i are thin items with small widths. R_i is special in L_n, because it may not appear in L_n. For any i, R_i is included in the sequence only if the online algorithm A satisfies some conditions which we will state in the following. For any i, s_i is a rectangle with small height and near-full width. After the online algorithm A packed the rectangle $P_1, s_1, \dots, s_i, \dots$ arrived online and may be packed either below P_1 or above P_1. When the first rectangle s_i(denoted by s_k) is packed above P_1, new s_i doesn't arrive. For any $i \ge 2$, F_i is a block item with the full width of 1. For convenience, s_k is also said to be F_1, i.e. $s_k = F_1$.

We can describe the sequences L_n (see Fig. 1) with

- $P_1 = 1, w(P_1) = \delta_0$
- $s_i = \theta + \varepsilon, w(s_i) = 1 - \delta_i$ for $i = 1, \dots, k$;
- $R_i = \frac{(\rho-1)(x_i+P_i+y_i)}{\rho} + \varepsilon$ for $i = 1, \dots, n-1$;
- $P_i = (x_{i-1} + P_{i-1} + y_{i-1}) + \varepsilon$ for $i = 2, \dots, n$;
- $F_2 = \max\{\min\{x_1 - \sum_{i=1}^{k-1} s_i, \frac{P_2}{\rho}\}, \min\{y_1, \frac{P_2}{\rho}\}, x_2\} + \varepsilon$;
- $F_i = \max\{\min\{y_{i-1}, \frac{P_i}{\rho}\}, F_{i-1}, x_i\} + \varepsilon$ for $i = 3, \dots, n-1$.

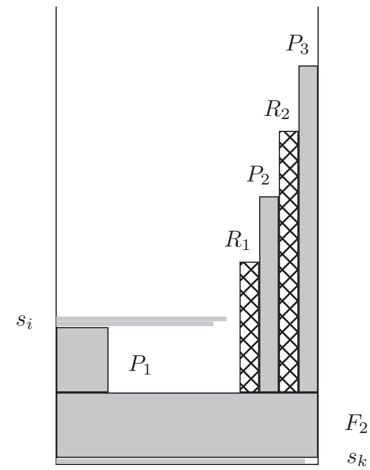

Fig. 2. An optimal packing.

The value ε is a small positive value and $\theta = \frac{-\rho^2+3\rho-1}{2(\rho-1)^2}$(Note that $\theta > 0$ if $\rho < (3+\sqrt{5})/2$). The value δ_0 is a positive number no more than $\frac{1}{2n-1}$ and $\delta_i = \frac{\delta_{i-1}}{4n}$ for $i = 1, \dots, k$. For $i \ge 2$, $w(R_{i-1}) = w(P_i) = 2\delta_k$ and $w(F_i) = 1$. The value x_1 denotes the distance between P_1 and the bottom of the strip, and x_i denotes the distance between F_{i-1} and P_i for $i = 2, \dots, n$. The value y_1 denotes the distance between P_1 and s_k ,and y_i denotes the distance between P_i and F_i for $i = 2, \dots, n-1$.

For $i = 1$, R_1 is included in the sequence if the following condition holds: either $x_1 - \sum_{i=1}^{k-1} s_i > \frac{P_2}{\rho}$ or $y_1 > \frac{P_2}{\rho}$(Note that the condition indicates that $x_1 + y_1 > \frac{P_2}{\rho}$, i.e. $x_1 + y_1 > \frac{P_1}{\rho-1}$.). For $i \ge 2$, R_i is included in the sequence if $x_i + y_i > \frac{P_i}{\rho-1}$. In the next section, we will show that R_i must be packed below F_i if the competitive ratio of A is less than $(3+\sqrt{5})/2$.

The sole function of the positive number ε is to ensure the structure of any online packing. For convenience, we assume that ε is small enough to be omitted from the analysis.

We now show that F_i should be packed above P_i for $i \ge 2$. For $i = 2$, if R_1 is included in the sequence which means either $x_1 - \sum_{i=1}^{k-1} s_i > \frac{P_2}{\rho}$ or $y_1 > \frac{P_2}{\rho}$, then $F_2 \ge \frac{P_2}{\rho}$. Thus $x_1 + P_1 + y_1 - R_1 = \frac{x_1+P_1+y_1}{\rho} = \frac{P_2}{\rho} \le F_2$ which means F_2 couldn't be packed below s_k. Since $F_2 \ge x_2$, the online algorithm A must pack F_2 above P_2. If R_1 is not included in the sequence, then $x_1 - (s_1 + \dots + s_{k-1}) \le \frac{P_2}{\rho}$ and $y_1 \le \frac{P_2}{\rho}$. Thus, $F_2 = \max\{x_1 - \sum_{i=1}^{k-1} s_i, y_1, x_2\}$ which means F_2 should be packed above P_2. For $i \ge 3$, if R_{i-1} is included in the sequence, let b_{i-1} be the distance between R_{i-1} and F_{i-1}, then $b_{i-1} \le x_{i-1} + P_{i-1} + y_{i-1} - R_{i-1} = \frac{x_{i-1}+P_{i-1}+y_{i-1}}{\rho} = \frac{P_i}{\rho}$. So $F_i = \max\{\min\{y_{i-1}, \frac{P_i}{\rho}\}, F_{i-1}, x_i\} \ge \max\{\min\{y_{i-1}, b_{i-1}\}, F_{i-1}, x_i\}$ which means F_i should be packed above P_i. If R_{i-1} is not included in the sequence, then $y_{i-1} \le \frac{P_i}{\rho}$(note that $x_{i-1} + y_{i-1} \le \frac{P_{i-1}}{\rho-1}$ indicates $y_{i-1} \le \frac{P_i}{\rho}$). So $F_i = \max\{\min\{y_{i-1}, \frac{P_i}{\rho}\}, F_{i-1}, x_i\} = \max\{y_{i-1}, F_{i-1}, x_i\}$ which also means F_i should be packed above P_i.

For the list $(J_1, \dots, J_i)$, let $A(J_i)$ denote the height of the packing by the algorithm A and $OPT(J_i)$ denote the height of the optimal off-line packing. It is not difficult to determine $OPT(J_j)$. We list them in the following (see Fig. 2):

- $OPT(P_1) = 1$;
- $OPT(s_i) = P_1 + \sum_{j=1}^{i} s_j$ for $i = 1, \dots, k$;
- $OPT(R_1) = P_1 + \sum_{j=1}^{k} s_j$ or $OPT(R_1) = R_1 + s_k$;
- $OPT(P_i) = P_i + \sum_{j=1}^{i-1} F_j$ for $i \ge 2$;
- $OPT(F_i) = P_i + \sum_{j=1}^{i} F_j$ for $i \ge 2$;
- $OPT(R_i) = R_i + \sum_{j=1}^{i} F_j$ for $i \ge 2$.