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Decision Support

Better than pre-commitment mean-variance portfolio allocation strategies: A semi-self-financing Hamilton–Jacobi–Bellman equation approach☆

D. M. Dang[a], P. A. Forsyth[b,*]

[a] School of Mathematics and Physics, The University of Queensland, St. Lucia, QLD 4072, Australia
[b] D. Cheriton School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada

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We generalize the idea of semi-self-financing strategies, originally discussed in Ehrbar (1990), and later formalized in Cui et al (2012), for the pre-commitment mean-variance (MV) optimal portfolio allocation problem. The proposed semi-self-financing strategies are built upon a numerical solution framework for Hamilton–Jacobi–Bellman equations, and can be readily employed in a very general setting, namely continuous or discrete re-balancing, jump-diffusions with finite activity, and realistic portfolio constraints. We show that if the portfolio wealth exceeds a threshold, an MV optimal strategy is to withdraw cash. These semi-self-financing strategies are generally non-unique. Numerical results confirming the superiority of the efficient frontiers produced by the strategies with positive cash withdrawals are presented. Tests based on estimation of parameters from historical time series show that the semi-self-financing strategy is robust to estimation ambiguities.

1. Introduction

1.1. Motivation

The mean-variance (MV) optimization criteria are popular for portfolio allocation problems, due to their intuitive nature (Basak & Chabakauri, 2010; Bielecki, Pliska, & Zhou, 2005; Leippold, Trojani, & Vanini, 2004; Li & Ng, 2000; Vigna, 2014; Wang & Forsyth, 2010; Zhou & Li, 2000). Under these criteria, risk is quantified by variance, so that investors aim to maximize the expected terminal wealth of their portfolios, given a risk level. Hence, the results can be easily interpreted in terms of the trade-off between the risk and the expected terminal portfolio wealth.

Mean-variance optimization typically yields *pre-commitment* strategies, which are time inconsistent (Basak & Chabakauri, 2010; Björk & Murgoci, 2010; Cui & Li, 2010; Cui, Li, Wang, & Zhu, 2012; Wang & Forsyth, 2011; 2012). However, it has been shown by Vigna (2014) that pre-commitment strategies can also be viewed as a target-based optimization which involves minimizing a quadratic loss function. Hence, these strategies are appropriate in the context of pension plan investment and insurance applications (Bauerle, 2005; Delong & Gerrard, 2007; Delong, Gerrard, & Haberman, 2008; Jose-Fombellida & Rincon-Zapatero, 2008). In fact, this phenomenon has been also discussed in the literature of MV hedging (see, for example Schweizer, 2010). In addition, it has also been pointed out that, in the context of optimal trade execution, the pre-commitment strategy optimizes trading efficiency as measured in practice (Almgren, 2012).

Previous work on pre-commitment MV optimal portfolio allocation has been dominated by the analytic (closed-form) approach. (See, for example, Bielecki et al., 2005; Li and Ng, 2000; Øksendal and Sulem, 2009; Zhou and Li, 2000, among many others.) However, this approach is not feasible when realistic constraints, such as no trading if insolvent and limited leverage, are imposed. In addition, from a risk management point of view, it is useful to model jumps in asset prices. In this case, it is necessary to impose a liquidation condition if the portfolio wealth jumps into the insolvent region. As a result, in these general situations, a fully numerical approach must be employed. It is important to highlight that realistic portfolio constraints and jumps are found to have pronounced effects on the efficient frontiers (Dang & Forsyth, 2014; Wang & Forsyth, 2010).

It is standard that MV strategies for the optimal portfolio allocation problem are *self-financing*, i.e. no exogenous infusion or withdrawal of cash are allowed under any circumstances. Central to our discussion is the concept of semi-self-financing. The term

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* Corresponding author. Tel.: 1 519 888 4567x4415.
E-mail addresses: duyminh.dang@uq.edu.au (D.M. Dang), paforsyt@uwaterloo.ca (P.A. Forsyth).

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semi-self-financing strategy is usually employed to refer to a strategy that exploits either exogenous infusion or withdrawal of cash, but not both. In our context, we strictly define a semi-self-financing strategy as a strategy that uses only non-negative cash withdrawals.

Ehrbar (1990) is possibly the first published work that touches upon the idea of semi-self-financing in the context of MV optimal portfolio allocation. As illustrated in Ehrbar (1990), even for a single-period model, it is possible to achieve a superior portfolio, i.e. a portfolio having the same standard derivation, but higher expected portfolio wealth, by *not* investing all of the initial wealth. It is further argued in Ehrbar (1990) that the self-financing strategy is unrealistic in the sense that it requires the investors "to invest *all* their money, even if the additional investments do *not* add to their utility". By withdrawing part of the initial portfolio, the investors can achieve superior results. It is also emphasized in Ehrbar (1990) that semi-self-financing strategies are "not only more straightforward", but also allow "investors to find better uses for the money they cannot invest".

Recently, the idea of semi-self-financing in the context of unconstrained pre-commitment MV optimal portfolio allocation is formalized in Cui and Li (2010) and Cui et al. (2012). In these papers, it is shown that, if the portfolio wealth exceeds a threshold at a re-balancing time, by removing a certain amount of cash from the portfolio, one can obtain a portfolio having the same expected wealth and standard deviation as the portfolio obtained by a self-financing MV optimal strategy. In addition, the investor receives a bonus in terms of a free cash flow.

1.2. Background and contributions

It is well-known that the MV optimal portfolio allocation problem is a multi-criteria optimization problem. Following a standard scalarization method for multi-criteria optimization, a single criterion can be formed by a positively weighted sum of the criteria (Yu, 1974). The resulting single-objective problem is referred to as the *MV scalarization problem*.

However, for MV optimization in general, and MV optimal portfolio allocation in particular, dynamic programming is not directly applicable to the MV scalarization problem, due to the presence of the variance term. To overcome this difficulty, an embedding technique is proposed in Li and Ng (2000) and Zhou and Li (2000) to embed the objective of the MV scalarization problem in a new single-objective optimization problem, namely the *embedded MV optimization problem*. Intuitively, this idea can be viewed as a quadratic target investment strategy (Vigna, 2014). Note that the embedding approach can be applied to general non-convex problems, in contrast to a Lagrange multiplier formulation (Li, Xhou, & Lim, 2002). Non-convex problems can arise if we consider non-linear effects, such as price impact (Tse, Forsyth, & Li, 2014).

Optimal solutions with respect to the embedded MV optimization problem can be obtained by solving an associated Hamilton–Jacobi–Bellman (HJB) equation. It has been established in Li and Ng (2000) and Zhou and Li (2000) that the MV scalarization optimal set is a *subset* of the embedded MV objective set. However, there may be points in the embedded MV objective set which are not in the MV scalarization optimal set. Methods for eliminating such spurious points are discussed in Dang, Forsyth, and Li (2015) and Tse et al. (2014). In the rest of the paper, to indicate the optimality of a strategy with respect to the MV scalarization problem and to the embedded MV optimization problem, we respectively use the terms *scalarization MV optimal/optimality* and *embedded MV optimal/optimality*.

The main contributions of the paper can be summarized as follows.

- In this paper, we generalize the idea of semi-self-financing strategies developed in Cui and Li (2010), Cui et al. (2012) and Ehrbar (1990) for the pre-commitment MV optimal portfolio allocation problem. Using the results in Dang and Forsyth (2014), Dang et al. (2015) and Tse et al. (2014), we formulate the embedded MV optimization problem in terms of the numerical solution of an HJB partial integro-differential equation (PIDE). Utilizing a fully numerical approach, it is straightforward to consider continuous or discrete re-balancing, jump-diffusions with finite activity, and realistic portfolio constraints. We determine an embedded MV optimal strategy over all possible semi-self-financing strategies which satisfy the constraints.
- We find certain cases where it can be proved that an embedded MV optimal semi-self-financing strategy involves withdrawing cash from the portfolio. These cases occur when the portfolio wealth exceeds the discounted optimal terminal wealth of the embedded problem. An embedded MV optimal strategy in this case is to (i) withdraw a specified amount of cash, and (ii) invest the remaining amount in the risk-free asset. However, embedded MV optimal semi-self-financing strategies are generally not unique.
- Using the numerical schemes discussed in Dang and Forsyth (2014) for the solution of the HJB equation, and using the results in Dang et al. (2015) and Tse et al. (2014), we can guarantee that scalarization MV optimal points, i.e. those that are on efficient frontiers, can be generated from embedded MV optimal points.
- We include several numerical examples to illustrate the superiority of strategies with positive cash withdrawals in a general setting where continuous and discrete re-balancing, realistic constraints, and jump-diffusions (with finite activity) are allowed.
- We estimate the jump diffusion parameters based on an 89 year time series of market return data. The jump parameter estimates are sensitive to the estimation method. However, the simulated investment results using the semi-self-financing mean-variance strategies are robust to estimated model parameter ambiguities.

2. Preliminaries

2.1. Underlying processes, allowable portfolios, and admissible sets

Since the portfolio can be either continuously or discretely re-balanced, we denote the set of discrete re-balancing times by

$$\mathcal{T}_M = \{t_0 = 0 < t_1 < \cdots < t_M = T\}.$$

Let

$$\mathcal{T} = \begin{cases} [0, T] & \text{continuous re-balancing,} \\ \mathcal{T}_M & \text{discrete re-balancing.} \end{cases}$$

Define $t^- = t - \epsilon$, where $\epsilon \to 0^+$, i.e. t^- is instant of time just before the (forward) time t, $t \in [0, T]$.

For simplicity, we assume that there are only two assets available in the financial market, namely a risky asset and a risk-free asset. We denote by $S(t)$ and $B(t)$ the *amounts* invested in risky and risk-free assets, respectively, at time t, $t \in [0, T]$. We denote by ξ the random number representing the jump multiplier. When a jump occurs, we have $S(t) = \xi S(t^-)$. As a specific example, in this paper, we consider ξ following a log-normal distribution $p(\xi)$ given by Merton (1976)

$$p(\xi) = \frac{1}{\sqrt{2\pi}\zeta\xi} \exp\left(-\frac{(\log(\xi) - \nu)^2}{2\zeta^2}\right), \tag{2.1}$$

with mean ν and standard deviation ζ, with $E[\xi] = \exp(\nu + \zeta^2/2)$, where $E[\cdot]$ denotes the expectation operator, and $\kappa = E[\xi] - 1$. In the absence of control, S follows the process:

$$\frac{dS(t)}{S(t^-)} = (\mu - \lambda\kappa)dt + \sigma dZ + d\left(\sum_{i=1}^{\pi_t} (\xi_i - 1)\right). \tag{2.2}$$