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Innovative Applications of O.R.

Asymmetric polygons with maximum area

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ABSTRACT

We say that a polygon inscribed in the circle is asymmetric if it contains no two antipodal points being the endpoints of a diameter. Given n diameters of a circle and a positive integer $k < n$, this paper addresses the problem of computing a maximum area asymmetric k-gon having as vertices $k < n$ endpoints of the given diameters. The study of this type of polygons is motivated by ethnomusiciological applications.

1. Introduction

There exist a number of significant links between music and Operations Research (OR) (Chew and Raphael, 2010). In fact, due its versatility in the mathematical modeling of optimization problems, operations researchers have tackled problems from many fields. Music (representation, analysis, composition, expressive performance, etc.) also presents rich application areas for OR techniques[1]. This article is an example of the opportunities that abound for operations researchers to make significant contributions in musicology.

Imagine that we are at a concert of salsa and we want to retain the intrinsic rhythmic pattern to dance appropriately. Then we should know the clave Son. The clave Son rhythm might be represented as the 16-bit binary sequence 1001001000101000 or, as usual for cyclic rhythms, by onsets represented as points on a circle as in Fig. 1(c). In this geometric setting, the clave Son is associated to a pentagon whose vertices are selected on a circular lattice of 16 points. So, the clave Son is represented by a special selection of k points on the circular lattice. This pattern has conquered our planet during the last half of the 20th century. But, what properties does this particular selection have? Notice that the polygon that represents the clave Son does not contain two antipodal points on the circle and moreover, it is easy to prove that this configuration is just the pentagon of maximum area without antipodal vertices (this later property produces a certain kind of asymmetry). Musicians have showed interest in finding similar patterns. Ethnomusicology is the discipline encompassing various approaches to the study of music that emphasize its cultural context. More specifically, Ethnomathematics is a domain consisting of the study of mathematical ideas shared by orally transmitted cultures. Such ideas are related to numeric, logic and spatial configurations (Ascher, 1998; Chemillier, 2002). Related to spacial configurations, the area is a useful measure of evenness of scales and rhythms in music theory (Arom et al., 1991; Rappaport, 2007). Moreover, many musical traditions all over the world have asymmetric rhythmic patterns. For instance, the Aka Pygmies rhythmic pattern in Fig. 1(b) has the so-called rhythmic oddity property discovered by (Arom et al., 1991). A rhythm has the *rhythmic oddity* property if when represented on a circle it does not contain two onsets (the black points in the Fig. (1) that lie diametrically opposite to each other. Thus, the property asserts that one cannot break the circle into two parts of equal length, whatever the chosen breaking point, as there is always one unit lacking on one side. This property produces a kind of perceptual asymmetry. The asymmetry of the pattern is to some extent intrinsic, in the sense that there exists no breaking point giving two parts of equal length. Note that the oddity property requires that the circle is divided into an even number of units. The notion of rhythmic oddity has received different mathematical treatments; see (Toussaint, 2010) for more details.

An algorithm for enumerating all the patterns satisfying the rhythmic oddity property has been proposed in (Chemillier and Truchet, 2003). Asymmetric rhythmics can also be found in the

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[1] We refer as example the special cluster of papers featured in the journal INFORMS Journal of Computing in 2006. http://pubsonline.informs.org/toc/ijoc/18/3

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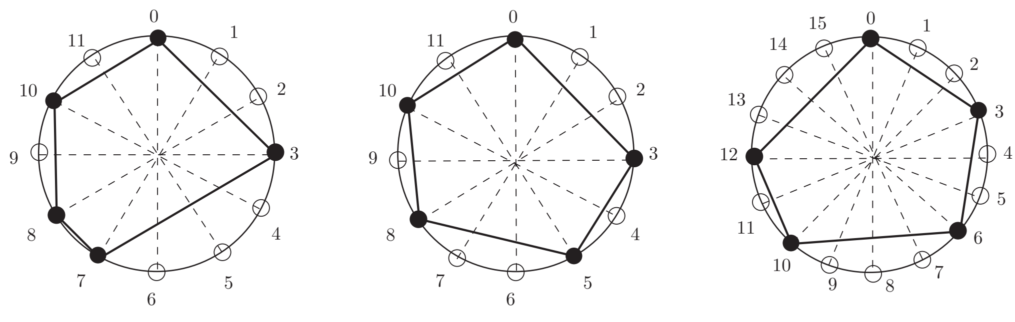

Fig. 1. (a) A flamenco rhythm in Spain, (b) a rhythm used in Central Africa, (c) the clave Son in Cuba.

flamenco music of Spain (Fig. 1(a)) and the clave Son in Cuba (Fig. 1(c)). See (Díaz-Báñez, 2013; Díaz-Báñez et al., 2004), and (Toussaint, 2005), for a detailed study on the preference of these rhythms in their cultural contexts.

Inspired by these ideas, we introduce the following geometric problem:

Problem 1.1. Given n diameters (not necessarily equally spaced) in a circle and a positive integer $k < n$, select k endpoints of these diameters, no two of the same diameter, in such a way that their convex hull defines a k-gon of maximum area.

Let us introduce some notation and related work. Let $(p_0, p'_0), \ldots, (p_{n-1}, p'_{n-1})$ be n diameters of a given circle. Let $S := \{p_0, p'_0, p_1, p'_1, \ldots, p_{n-1}, p'_{n-1}\}$ be an *antipodal set* containing the $2n$ endpoints of the given diameters. A *sub-polygon* (of S) is a convex polygon whose vertex set is a subset of S. An *antipodal polygon* (on S) is a sub-polygon whose vertex set contains exactly one endpoint from each diameter (p_i, p'_i) of S (Aichholzer et al., 2015). An *asymmetric polygon* is a sub-polygon that contains no diameter. Therefore, an antipodal polygon is also an asymmetric polygon. Aichholzer *et al.* proved that an antipodal polygon of maximum area can be found in $\Theta(n)$-time (Aichholzer et al., 2015). The linear time algorithms they proposed are strongly based on a simple characterization for the extremal antipodal polygons. Namely, that an antipodal polygon of maximum area has an alternating configuration (Aichholzer et al., 2015).

The problem however, is significantly different if we ask for an asymmetric k-gon of maximum area with $k < n$ vertices in S. It is not difficult to come up with examples for which the simple characterization stated above does not hold if $k < n$. Aichholzer *et al.*presented an $O(n^{n-k+1})$-time algorithm to compute an antipodal k-gon of maximum area. However, the existence of a polynomial time algorithm to solve this problem was left as an open question (Aichholzer et al., 2015).

In this paper, we answer this question in the affirmative. We distinguish two cases: if we are given a circular lattice with an antipodal set of $2n$ points (induced by n evenly spaced diameters), we show how to solve Problem 1.1 in constant time by providing a characterization of the solution. Otherwise, if the diameters are given in a general configuration, we show that the problem can be solved in $O(kn^4)$-time using dynamic programming.

The problem studied here is related to other optimization problems in operations research and mathematics. A similar problem consists of finding the unit-diameter polygon of maximal area. The first progress was made in (Reinhardt, 1922) where the optimality of the regular polygon was proved for polygons with odd number of sides. The problem of determining the largest area a plane hexagon of unit diameter can have was solved in (Graham, 1975), the octagon was solved in (Audet et al., 2002) and finally, a family of nearly optimal polygons is given in (Mossinghoff, 2006).

In computational geometry, efficient algorithms have been proposed for computing extremal polygons with respect to several different properties (Boyce et al., 1985). Moreover, the so-called stabbing or transversal problems (see for instance Arkin et al., 2011) belong to the same family that our problem. Recently, it has been proved that the following problem is NP-hard (Díaz-Báñez et al., 2015): given a set S of line segments, compute the minimum or maximum area (perimeter) polygon P such that P stabs S, that is, at least one of the two endpoints of every segment $s \in S$ is contained in P. In operations research, global optimization techniques have been extensively studied to find convex polygons maximizing a given parameter (Audet et al., 2007).

Now, let us consider a different interpretation of the problem, let P be the convex hull of the given diameters. In this case, the solution to Problem 1.1 can be interpreted as the asymmetric polygon with $k < n$ vertices that has its area closest to that of P. Therefore, this problem is related to the approximation of convex sets. In this setting, the best inner approximation of any convex set by a symmetric set is studied in (Lassak, 2002). Moreover, if we consider the “distance” to a symmetric set to be a measure of its symmetry (Grunbaum, 1963), then our solution to Problem 1.1 provides the best approximation of a convex polygon inscribed in a circle by asymmetric sub-polygons.

Finally, it is worth mentioning that although there exist applications of operations research to music theory (see for example Schell, 2002), the research in music has illuminated problems that are appealing, nontrivial, and, in some cases, connected to deep mathematical questions. The problem introduced in this paper could be an example.

The remainder of the paper is organized as follows. In Section 2 we consider the constrained version of Problem 1.1 in which the endpoints of the diameters are evenly spaced on the circle. In this case, we provide a characterization of the maximum k-gon which yields a constant time algorithm to solve Problem 1.1. The general version of this problem where the endpoints of the diameters are distributed anywhere on the circle is studied in Section 3. In this case, we show that Problem 1.1 can be solved in polynomial time using dynamic programming.

2. Evenly spaced diameters

In this section, we assume that $S = \{p_0, p'_0, p_1, p'_1, \ldots, p_{n-1}, p'_{n-1}\}$ is the set of endpoints of n evenly spaced diameters on a unit circle. That is, S partition the circle into $2n$ arcs of equal length. A k-gon Q_k with vertices $q_0, \ldots, q_{k-1}$ in S can be encoded by a sequence $< a_0, \ldots, a_{k-1} >$, where a_i is the number of arcs between the vertices q_i and q_{i+1}, $0 \le i < k$, $q_k = q_0$. For example, the clave Son in Fig. 1(c) can be encoded by the sequence $< 3, 3, 4, 2, 4 >$. In music theory, this sequence is called the interval vector (or full-interval vector) (McCartin, 1998). As with rhythmic patterns, we assume that