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Continuous Optimization

On the solution of multidimensional convex separable continuous knapsack problem with bounded variables

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A R T I C L E I N F O

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A B S T R A C T

A minimization problem with a convex separable objective function subject to linear equality constraints and box constraints (bounds on the variables) is considered. Necessary and sufficient optimality condition is proved for a feasible solution to be an optimal solution to this problem. Primal-dual analysis is also included. Examples of some convex separable objective functions for the considered problem are presented.

1. Introduction

Consider the following convex separable program $(C_m^=)$

$$\min \left\{ c(\mathbf{x}) = \sum_{j \in J} c_j(x_j) \right\} \tag{1}$$

subject to

$$\sum_{j \in J} d_{ij} x_j = \alpha_i, \quad i = 1, \dots, m \tag{2}$$

$$a_j \le x_j \le b_j, \quad j \in J, \tag{3}$$

where $c_j(x_j)$, $j \in J$, are differentiable convex functions, defined on the open convex sets X_j, $j \in J$, respectively, $d_{ij} > 0$ for every $i = 1, \dots, m$ and $j \in J$; $\alpha_i \in \mathbb{R}$, $i = 1, \dots, m$; $a_j, b_j \in \mathbb{R}$, $j \in J$; $\mathbf{x} = (x_j)_{j \in J}$, where $J \equiv \{1, ..., n\}$.

The feasible region X, defined by constraints (2) and (3), is a convex set because it is an intersection of the convex set (2), defined by m hyperplanes, and the box (3) of dimension $|J| = n$.

Feasible region of the form (2)–(3) is a polytope and it is known as the *(multiple) knapsack polytope*.

Problem $(C_m^=)$ and related problems arise in many cases, for example, in production planning and scheduling (Bitran & Hax, 1981), in allocation of resources (Bitran & Hax, 1981; Zipkin, 1980), in allocation of effort resources among competing activities (Luss & Gupta, 1975), in the theory of search (Charnes & Cooper, 1958), in subgradient optimization (Held, Wolfe, & Crowder, 1974), in facility location problems (Stefanov, 2000), in the implementation of various projection methods when the feasible region is of the form (2)–(3) (Stefanov, 2000; 2001; 2004), etc. That is why, characterization of the optimal solution of problem $(C_m^=)$ as well as efficient algorithms for solving such problems will be very useful.

Related problems and methods for solving them are considered in the papers listed in References. The solution of knapsack problems with arbitrary convex or concave objective functions is studied in Bitran and Hax (1981), Luss and Gupta (1975), Moré and Vavasis (1991), Zipkin (1980), etc. Quadratic knapsack problems and related to them are studied in Brucker (1984), Pardalos, Ye, and Han (1991), Robinson, Jiang, and Lerme (1992), etc. A branch and bound algorithm for separable concave programming is proposed in Hong-gang Xue, Cheng-xian Xu, and Feng-min Xu (2004). Algorithms for the case of convex quadratic objective function are proposed in Brucker (1984), Dussault, Ferland, and Lemaire (1986), Helgason, Kennington, and Lall (1980), etc. Algorithms for bound constrained quadratic programming problems are proposed in Dembo and Tulowitzki (1983), Moré and Toraldo (1989), Pardalos and Kovoor (1990). A polynomial time algorithm for the resource allocation problem with a convex objective function and nonnegative integer variables is suggested in Katoh, Ibaraki, and Mine (1979). Surrogate upper bound sets for bi-objective bi-dimensional binary knapsack problems are studied in Cerqueus, Przybylski, and Gandibleux (2015). Valid inequalities, cutting planes and integrality of the knapsack polytope, which is the feasible region of the problem under consideration, are considered, for example, in Stefanov (1998; 2011), etc. Well-posedness and primal-dual analysis of convex separable optimization problems of the considered form are studied in Stefanov (2013b). Methods for solving variational inequalities over box constrained feasible regions, defined by (3), are proposed, for example, in Stefanov (2002; 2007; 2013a), etc. Iterative methods of nondifferentiable optimization are

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applied to some separable problems of approximation theory in Stefanov (2014), etc.

This paper is devoted to formulation and proof of a necessary and sufficient optimality condition (characterization) of optimal solution to problem ($C_m^=$). Rest of the paper is organized as follows. In Section 2, a necessary and sufficient condition (characterization theorem) for a feasible solution to be an optimal solution to problem ($C_m^=$) is proved (Theorem 1). In Section 3, primal-dual analysis of the proposed approach is included. In Section 4, some strictly convex separable functions $c_j(x_j), j \in J$, important for practical problems like ($C_m^=$), are presented. In Section 5, open problem for future work is formulated.

2. Main result

2.1. Characterization theorem

Suppose that the following assumptions are satisfied.

(A1) $a_j \le b_j$ for all $j \in J$. If $a_k = b_k$ for some $k \in J$, then the value $x_k := a_k = b_k$ is determined a priori.

(A2) $\sum_{j\in J} d_{ij}a_j \le \alpha_i \le \sum_{j\in J} d_{ij}b_j, i = 1, \dots, m$. Otherwise the constraints (2) and (3) are inconsistent and the feasible region $X = \emptyset$.

The Lagrangian for problem ($C_m^=$) is

$$L(\mathbf{x}, \mathbf{u}, \mathbf{v}, \lambda) = \sum_{j\in J} c_j(x_j) + \sum_{i=1}^{m} \lambda_i \left(\sum_{j\in J} d_{ij}x_j - \alpha_i \right) + \sum_{j\in J} u_j(a_j - x_j) + \sum_{j\in J} v_j(x_j - b_j), \tag{4}$$

where $\lambda \in \mathbb{R}^m$; $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ consists of all vectors with n real nonnegative components.

The Karush–Kuhn–Tucker (KKT) necessary and sufficient optimality conditions for the minimum solution $\mathbf{x}^* = (x_j^*)_{j\in J}$ to problem ($C_m^=$) are

$$c_j'(x_j^*) + \sum_{i=1}^{m} \lambda_i d_{ij} - u_j + v_j = 0, \quad j \in J, \tag{5}$$

$$u_j(a_j - x_j^*) = 0, \quad j \in J, \tag{6}$$

$$v_j(x_j^* - b_j) = 0, \quad j \in J, \tag{7}$$

$$\sum_{j\in J} d_{ij}x_j^* = \alpha_i, \quad i = 1, \dots, m, \tag{8}$$

$$a_j \le x_j^* \le b_j, \quad j \in J, \tag{9}$$

$$\lambda_i \in \mathbb{R}^1, \quad i = 1, \dots, m, \quad u_j \in \mathbb{R}^1_+, v_j \in \mathbb{R}^1_+, \quad j \in J, \tag{10}$$

where $\lambda_i, i = 1, \dots, m$; $u_j, v_j, j \in J$, are the Lagrange multipliers, associated with the constraints (2), $a_j \le x_j$, $x_j \le b_j$, $j \in J$, respectively. If $a_j = -\infty$ or $b_j = +\infty$ for some $j \in J$, we do not consider the corresponding KKT condition (6) (KKT condition (7), respectively) and multiplier u_j (multiplier v_j, respectively).

The following Theorem 1 gives necessary and sufficient optimality condition (characterization) of the optimal solution to problem ($C_m^=$).

Theorem 1. *(Characterization of the optimal solution to problem ($C_m^=$)) Let assumptions (A1)and(A2) be satisfied. A feasible solution $\mathbf{x}^* = (x_j^*)_{j\in J} \in X$ is an optimal solution to problem ($C_m^=$) if and only if there exist $\lambda_i \in \mathbb{R}, i = 1, \dots, m$, such that*

$$x_j^* = a_j, \quad j \in J_a^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -c_j'(a_j) \le \sum_{i=1}^{m} \lambda_i d_{ij} \right\} \tag{11}$$

$$x_j^* = b_j, \quad j \in J_b^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -c_j'(b_j) \ge \sum_{i=1}^{m} \lambda_i d_{ij} \right\} \tag{12}$$

$$x_j^* : -c_j'(x_j^*) = \sum_{i=1}^{m} \lambda_i d_{ij},$$

$$j \in J^\lambda \stackrel{\text{def}}{=} \left\{ j \in J : -c_j'(b_j) \le \sum_{i=1}^{m} \lambda_i d_{ij} \le -c_j'(a_j) \right\}. \tag{13}$$

When convex function $c_j(x_j), j \in J$, are strictly convex, then inequalities, defining J^λ in (13), are strict.

Proof. *Necessity*. Let $\mathbf{x}^* = (x_j^*)_{j\in J}$ be an optimal solution to problem ($C_m^=$). According to KKT theorem, there exist constants $\lambda_i, i = 1, \dots, m$; $u_j, v_j, j \in J$, such that KKT conditions (5)–(10) are satisfied.

(a) If $x_j^* = a_j$, then $u_j \ge 0, v_j = 0$ according to KKT conditions (10) and (7), respectively. Therefore, (5) implies

$$c_j'(x_j^*) \equiv c_j'(a_j) = u_j - \sum_{i=1}^{m} \lambda_i d_{ij} \ge - \sum_{i=1}^{m} \lambda_i d_{ij}.$$

Multiplying this inequality by (-1), we obtain

$$-c_j'(a_j) \le \sum_{i=1}^{m} \lambda_i d_{ij}.$$

(b) If $x_j^* = b_j$, then $u_j = 0, v_j \ge 0$ according to KKT conditions (6) and (10), respectively. Therefore, (5) implies

$$c_j'(x_j^*) \equiv c_j'(b_j) = -v_j - \sum_{i=1}^{m} \lambda_i d_{ij} \le - \sum_{i=1}^{m} \lambda_i d_{ij}.$$

Multiplying this inequality by (-1), we get

$$-c_j'(b_j) \ge \sum_{i=1}^{m} \lambda_i d_{ij}.$$

(c) If $a_j < x_j^* < b_j$, then $u_j = v_j = 0$ according to KKT conditions (6) and (7), respectively. Therefore, (5) implies

$$-c_j'(x_j^*) = \sum_{i=1}^{m} \lambda_i d_{ij}.$$

Using that $a_j < x_j^* < b_j$ and $c_j(x_j), j \in J$, are convex functions, it follows that $c_j'(a_j) \le c_j'(x_j^*) \le c_j'(b_j)$, that is, in case (c) we have

$$-c_j'(b_j) \le \sum_{i=1}^{m} \lambda_i d_{ij} \le -c_j'(a_j).$$

When convex function $c_j(x_j), j \in J$, are *strictly* convex, these inequalities are strict.

In order to describe cases (a), (b), (c), we introduce the index sets $J_a^\lambda, J_b^\lambda, J^\lambda$, defined by (11), (12), and (13), respectively. It is obvious that $J_a^\lambda \cup J_b^\lambda \cup J^\lambda = J$. The “necessity” part of Theorem 1 is proved.

Sufficiency. Conversely, let $\mathbf{x}^* \in X$ and components of $\mathbf{x}^*$ satisfy (11), (12), (13).

Set:

$$\sum_{i=1}^{m} \lambda_i d_{ij} = -c_j'(x_j^*), \quad u_j = v_j = 0 \quad \text{for } j \in J^\lambda;$$

$$u_j = c_j'(a_j) + \sum_{i=1}^{m} \lambda_i d_{ij} \quad (\ge 0 \text{ according to definition of } J_a^\lambda),$$

$$v_j = 0 \quad \text{for } j \in J_a^\lambda;$$

$$u_j = 0, \quad v_j = -c_j'(b_j) - \sum_{i=1}^{m} \lambda_i d_{ij} \quad (\ge 0 \text{ according to definition of } J_b^\lambda)$$

for $j \in J_b^\lambda$.