



Stochastics and Statistics

Convergence results for patchwork copulas



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ARTICLE INFO

Article history:

Received 9 August 2014

Accepted 11 June 2015

Available online 19 June 2015

Keywords:

Copula

Checkerboard copula

Convergence of copulas

ABSTRACT

We present a general construction that allows to extend a given subcopula to a copula in such a way that the extension is affine on some specific segments of the copula domain. This construction is hence applied to provide convergence theorems for approximating a copula in strong convergence and in D_1 -metric (related to the Markov kernel representation of a copula).

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1. Introduction

Copulas are multivariate probability distribution functions whose univariate margins are uniformly distributed on the unit interval $[0, 1]$. They represent one of the building blocks of (modern) multivariate analysis since Sklar showed that the probability law of any random vector $\mathbf{X}$ can be expressed as a composition of the distribution functions of all one-dimensional margins and a suitable copula (Faugeras, 2013; Rüschendorf, 2009; Sklar, 1959). This fact has provided to be useful in various applications including decision science (see, e.g., Abbas, 2013) and reliability theory (see, e.g., Gupta, Misra, & Kumar, 2015; Navarro, Pellerey, & Crescenzo, 2015; Rychlik, 2010).

Now, while the copula associated with $\mathbf{X}$ is unique when the margins are continuous, in the non-continuous case we should take into account that various copulas can be associated with $\mathbf{X}$. In fact, every copula associated with a non-continuous random vector $\mathbf{X}$ is uniquely determined only on a Borel set $A \subset [0, 1]^d$. This fact poses the natural question of how it is possible to construct a copula given some partial information about the values that it assumes on specific subsets of the domain $[0, 1]^d$ (see, for instance, Carley & Taylor, 2002). In this respect, one of the most common extension procedures is given by the multilinear interpolation (or checkerboard construction), originally proposed by Schweizer and Sklar (1974) and, further, developed by Carley and Taylor (2002), and Genest, Nešlehová, and Rémillard

(2014). In fact, as shown by Genest and Nešlehová (2007) and Nešlehová (2007), this extension plays a central role in characterizing dependence concepts for discrete random vectors.

The aim of this paper is to contribute to the development of extension procedures that include previously introduced methodologies (e.g., checkerboard), but are naturally presented in a high-dimensional framework. Our starting point is represented by the fact that many of these constructions can be generally classified as patchwork copulas (Durante, Fernández-Sánchez, & Sempí, 2013; Durante, Saminger-Platz, & Sarkoci, 2009). Roughly speaking, a patchwork copula is any copula that is obtained by modifying the probability measure associated with a given copula C_B in several given subsets of its domain in a way that is specified by other copulas $C_1, \dots, C_n$. Following this approach, we present hence a general construction that allows to extend a specific subcopula to a copula (Section 2). Moreover, we also provide some convergence results in order to check how these extensions approximate (in different metrics) a given copula (Section 3).

2. Basic definitions and properties

Throughout the paper, the natural number $d \geq 2$ will denote the dimension, $\mathbb{I} := [0, 1]$, $\mathcal{B}(\mathbb{I}^d)$ the Borel- σ -field on $\mathbb{I}^d$, and λ_d the Lebesgue measure on $\mathcal{B}(\mathbb{I}^d)$.

For basic definitions and properties of measure theory (including Disintegration Theorem), we refer to Ash (2000) and Billingsley (1995). For basic definitions and properties of copulas, we refer to Durante and Sempí (2015) and Nelsen (2006).

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We recall that the space of copulas is in one-to-one correspondence with the space of all probability measures μ on $\mathcal{B}(\mathbb{I}^d)$ that are d -fold stochastic, i.e. such that all one-dimensional marginals μ^{π_i} coincide with λ , the Lebesgue measure on $\mathcal{B}(\mathbb{I})$, whereby μ^{π_i} denotes the push-forward of μ under the i th canonical projection π_i .

In order to introduce our concepts, we need some preliminary notations.

Given a fixed natural number $d \geq 2$, let $n_1, \dots, n_d$ be d natural numbers, and, for every $k \in \{1, \dots, d\}$, let $a_0^k, a_1^k, \dots, a_{n_k}^k$ be numbers in $\mathbb{I}$ such that

$$0 = a_0^k < a_1^k < \dots < a_{n_k}^k = 1.$$

Consider the mesh

$$D := \times_{k=1}^d \{a_0^k, a_1^k, \dots, a_{n_k}^k\} = \times_{k=1}^d I_k,$$

which divides $\mathbb{I}^d$ into $\prod_{i=1}^d n_i$ d -boxes

$$B_{(i_1, i_2, \dots, i_d)} := [a_{i_1}^1, a_{i_1+1}^1] \times [a_{i_2}^2, a_{i_2+1}^2] \cdots \times [a_{i_d}^d, a_{i_d+1}^d], \quad (1)$$

where for every $k \in \{1, 2, \dots, d\}$, $i_k \in \{0, \dots, n_k - 1\}$. The norm of the mesh D is defined by $\|D\| = \max_{k, i_k} (a_{i_k+1}^k - a_{i_k}^k)$. A mesh D' is a refinement of D when $D \subseteq D'$.

Moreover, for every $k \in \{1, 2, \dots, d\}$ and every $i_k \in \{0, 1, \dots, n_k - 1\}$, consider the function $f_{i_k}^k : \mathbb{I} \rightarrow \mathbb{I}$ given by

$$f_{i_k}^k(x) := \begin{cases} 0, & x < a_{i_k}^k, \\ \frac{x - a_{i_k}^k}{a_{i_k+1}^k - a_{i_k}^k}, & a_{i_k}^k \leq x \leq a_{i_k+1}^k, \\ 1, & x > a_{i_k+1}^k. \end{cases} \quad (2)$$

For the multi-index $\mathbf{i} = (i_1, \dots, i_d) \in \mathcal{I} = \{0, 1, \dots, n_1 - 1\} \times \dots \times \{0, 1, \dots, n_d - 1\}$, we define $f_{\mathbf{i}} : \mathbb{I}^d \rightarrow \mathbb{I}^d$ via

$$f_{\mathbf{i}}(\mathbf{u}) = (f_{i_1}^1(u_1), \dots, f_{i_d}^d(u_d)).$$

We are now able to give the following definition.

Definition 2.1. Let S be a subcopula with domain D . For each $\mathbf{i} \in \mathcal{I}$, consider a measure $\mu_{\mathbf{i}}$ on $\mathcal{B}(\mathbb{I}^d)$. Define the measure μ on $\mathcal{B}(\mathbb{I}^d)$ such that for every Borel set $B \in \mathcal{B}(\mathbb{I}^d)$, one has

$$\mu(B) = \sum_{\mathbf{i} \in \mathcal{I}} \beta_{\mathbf{i}} \mu_{\mathbf{i}}(f_{\mathbf{i}}(B \cap B_{\mathbf{i}})), \quad (3)$$

where $\beta_{\mathbf{i}}$ is equal to the S -volume of $B_{\mathbf{i}}$, i.e. $\beta_{\mathbf{i}} = V_S(B_{\mathbf{i}})$.

Notice that the set-function μ is a bona fide measure because each $\mu_{\mathbf{i}}$ is a measure. Moreover, the following result holds.

Proposition 2.1. Under the assumptions of Definition 2.1, suppose that, for each $\mathbf{i} \in \mathcal{I}$, $\mu_{\mathbf{i}}$ is a probability measure such that $\mu_{\mathbf{i}}([0, 1]^d) = 1$. Then μ defined by (3) is a probability measure. Moreover, if each $\mu_{\mathbf{i}}$ is d -fold stochastic, so is μ .

Proof. Obviously, μ is a measure and, since each $\mu_{\mathbf{i}}$ has no mass on the border of $\mathbb{I}^d$, it follows easily that the total mass of μ is 1.

Moreover, consider the case when each $\mu_{\mathbf{i}}$ is d -fold stochastic. Let $u_1 \in \mathbb{I}$ such that $a_{i_1}^1 \leq u_1 \leq a_{i_1+1}^1$. Let

$$B = [0, u_1] \times \mathbb{I}^{d-1} = ([0, a_{i_1}^1] \times \mathbb{I}^{d-1}) \cup (\cup_{\mathbf{j} \in \mathcal{J}'} (B_{\mathbf{j}} \cap B)),$$

where the set $\mathcal{J}'$ is formed by all indices $\mathbf{j} = (j_1, \dots, j_d)$ with $j_1 = i_1$. Thus, it holds

$$\begin{aligned} \mu([0, u_1] \times \mathbb{I}^{d-1}) &= S(a_{i_1}^1, 1, \dots, 1) + \sum_{\mathbf{j} \in \mathcal{J}'} \beta_{\mathbf{j}} \frac{u_1 - a_{i_1}^1}{a_{i_1+1}^1 - a_{i_1}^1} \\ &= a_{i_1}^1 + \frac{u_1 - a_{i_1}^1}{a_{i_1+1}^1 - a_{i_1}^1} (a_{i_1+1}^1 - a_{i_1}^1) = u_1, \end{aligned}$$

because, for every $\mathbf{j} \in \mathcal{J}'$,

$$\mu_{\mathbf{j}}(f_{\mathbf{j}}(B \cap B_{\mathbf{j}})) = \mu_{\mathbf{j}}\left(\left[0, \frac{u_1 - a_{i_1}^1}{a_{i_1+1}^1 - a_{i_1}^1}\right] \times \mathbb{I}^{d-1}\right) = \frac{u_1 - a_{i_1}^1}{a_{i_1+1}^1 - a_{i_1}^1}.$$

By repeating the above procedure for each coordinate, we have the desired assertion. $\square$

Roughly speaking, the measure μ defined by (3) is obtained by putting together several measures, each of them acting in one single set of a given partition of the domain $\mathbb{I}^d$. The only constraint is provided by the subcopula S , which fixes the value of the measure at each box whose vertices belong to the mesh D .

From now on, we are interested in measures μ of type (3) that are generated by d -fold stochastic measures $(\mu_{\mathbf{i}})_{\mathbf{i} \in \mathcal{I}}$. In this case, the explicit expression of the copula C related to μ is given by the following result.

Proposition 2.2. Under the assumptions of Definition 2.1, suppose that, for each $\mathbf{i} \in \mathcal{I}$, $\mu_{\mathbf{i}}$ is a d -fold stochastic measure. Moreover, $C_{\mathbf{i}}$ denotes the copula associated to the measure $\mu_{\mathbf{i}}$. Then the copula associated to the measure μ can be written, for every $\mathbf{u} \in B_{\mathbf{i}}$ with $\mathbf{i} \in \mathcal{I}$, by

$$C(\mathbf{u}) = S(a_{i_1}^1, \dots, a_{i_d}^d) + \sum_{\mathbf{j} \in \mathcal{J}_i} \beta_{\mathbf{j}} C_{\mathbf{j}}(f_{\mathbf{j}}(\mathbf{u})). \quad (4)$$

Here the set $\mathcal{J}_i$ is formed by all indices $\mathbf{j} = (j_1, \dots, j_d)$ with $j_k \leq i_k$ for every $k = 1, 2, \dots, d$, and $j_k = i_k$ for at least one index k .

Proof. Let $\mathbf{u} \in B_{\mathbf{i}}$. Then $[0, \mathbf{u}]$ can be decomposed into the following union of boxes whose interiors are disjoint:

$$[0, \mathbf{u}] = ([0, a_{i_1}^1] \times \dots \times [0, a_{i_d}^d]) \cup (\cup_{\mathbf{j} \in \mathcal{J}_i} (B_{\mathbf{j}} \cap [0, \mathbf{u}])). \quad (5)$$

Thus, we have

$$\begin{aligned} C(\mathbf{u}) &= \mu([0, a_{i_1}^1] \times \dots \times [0, a_{i_d}^d]) + \sum_{\mathbf{j} \in \mathcal{J}_i} \beta_{\mathbf{j}} \mu_{\mathbf{j}}(f_{\mathbf{j}}([a_{j_1}^1, u_1] \\ &\quad \times \dots \times [a_{j_d}^d, u_d])) = S(a_{i_1}^1, \dots, a_{i_d}^d) + \sum_{\mathbf{j} \in \mathcal{J}_i} \beta_{\mathbf{j}} C_{\mathbf{j}}(f_{\mathbf{j}}(\mathbf{u})), \end{aligned}$$

which is the desired assertion. $\square$

It easily follows that $C = S$ on the mesh D . Moreover, let $\mathbf{u} = (u_1, a_{i_2}^2, \dots, a_{i_d}^d)$ be in $B_{\mathbf{i}}$. Let $[0, \mathbf{u}]$ be decomposed as in (5). If $\mathbf{j} \in \mathcal{J}_i^1$, then

$$\beta_{\mathbf{j}} C_{\mathbf{j}}(f_{j_1}^1(u_1), f_{j_2}^2(a_{i_2}^2), \dots, f_{j_d}^d(a_{i_d}^d)) = \beta_{\mathbf{j}} \frac{u_1 - a_{i_1}^1}{a_{i_1+1}^1 - a_{i_1}^1}.$$

Hence

$$\begin{aligned} C(\mathbf{u}) - C(a_{i_1}^1, a_{i_2}^2, \dots, a_{i_d}^d) &= \mu([0, \mathbf{u}]) - S(a_{i_1}^1, a_{i_2}^2, \dots, a_{i_d}^d) \\ &= \frac{u_1 - a_{i_1}^1}{a_{i_1+1}^1 - a_{i_1}^1} \sum_{\mathbf{j} \in \mathcal{J}_i^1} \beta_{\mathbf{j}} = \frac{u_1 - a_{i_1}^1}{a_{i_1+1}^1 - a_{i_1}^1} \mu([a_{i_1}^1, a_{i_1+1}^1] \times [0, a_{i_2}^2] \\ &\quad \times \dots \times [0, a_{i_d}^d]) = \frac{u_1 - a_{i_1}^1}{a_{i_1+1}^1 - a_{i_1}^1} (S(a_{i_1+1}^1, a_{i_2}^2, \dots, a_{i_d}^d) \\ &\quad - S(a_{i_1}^1, a_{i_2}^2, \dots, a_{i_d}^d)). \end{aligned}$$

By repeating the above procedure for each coordinate, it follows that the extension of the subcopula S to the copula C provided by (4) is linear along segments joining two points in the mesh D such that $(d-1)$ of their components are equal.

For such a reason, the copula C defined by (4) is called the *L-extension* of S by means of $(C_{\mathbf{i}})_{\mathbf{i} \in \mathcal{I}}$, where the prefix “L” indicates that such extension is linear on specific segments of the copula domain. We write $C = \langle D, C_{\mathbf{i}} \rangle_{\mathbf{i} \in \mathcal{I}}^S$ or, with the measure notation, $\mu = \langle D, \mu_{\mathbf{i}} \rangle_{\mathbf{i} \in \mathcal{I}}^S$.

Let $C = \langle D, C_{\mathbf{i}} \rangle_{\mathbf{i} \in \mathcal{I}}^S$. If every $C_{\mathbf{i}}$ is absolutely continuous (respectively, singular), then C is absolutely continuous (respectively, singular). In particular, if every $C_{\mathbf{i}}$ equals the independence copula Π_d , C

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