European Journal of Operational Research 247 (2015) 610–617

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Comparative statics effects independent of the utility function. When do we act the same way under risk?

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ARTICLE INFO

Article history:
Received 30 May 2014
Accepted 22 June 2015
Available online 25 June 2015

Keywords:
Utility theory
Comparative statics
Risk aversion
Portfolio choice
Firm behavior

ABSTRACT

To date, comparative statics effects analyzed in the context of expected utility depend on the decision-maker's utility function. This paper presents a methodology that allows comparative statics effects to be obtained independent of the utility function for a general family of problems. In other words, we provide a set of conditions under which all decision-makers act the same way. Such a methodology can be applied to relevant problems in the related literature, such as the portfolio choice and the competitive firm under price uncertainty (with or without the existence of a forward/futures market). With regards to such specific cases, we show and analyze the most relevant effects produced by the variation of two or more parameters, regardless of the decision-maker's preferences and attitude towards risk. Some of these effects are illustrated numerically.

1. Introduction

In the literature on decision-making there are several authors who model choice under risk by relying on the assumption of expected utility (EU) maximization. Amongst these authors, we can cite the seminal papers by Arrow (1963; 1965; 1971), on the optimal portfolio composition; Sandmo (1971), on the production-related decisions made by the competitive firm under uncertainty; and Feder (1977), on the problem of hiring under uncertainty. In Lippman and McCall (1981), we can find a good sample of classic models based on this same assumption. We can also cite more recent works such as those by Eeckhoudt, Gollier, and Schlesinger (1995) and Sévi (2010) on the single-period newsvendor problem; Choi and Ruszczyński (2011), on the multi-product newsvendor problem; Müller (2000), on the optimal stopping problem; Dalal and Alghalith (2009), on the firm under multiple uncertainty sources; Alghalith (2010), on methodological studies; Andersen and Nielsen (2013), on sports economics; and Rodríguez-Puerta and Álvarez-López (2014), on optimal allocation.

One of the studies that can be conducted on the above-mentioned models is the comparative statics analysis. This method, proposed by Samuelson (1941), may be approached in a simplified way (for one variable) by considering an unknown variable x whose equilibrium value is to be determined for preassigned values of a parameter y. We assume the following continuously differentiable implicit relation involving the unknown variable and the parameter:

$$\phi(x, y) = 0. \tag{1}$$

If, for a given value of $y = y_0$, there exists a solution x^* to (1), and if $\phi_x(x^*, y_0) \neq 0$, then by the implicit-function theorem[1] there exists a continuously differentiable function ψ in a sufficiently small neighborhood I of y_0 such that $x^* = \psi(y_0)$ and $\phi(\psi(y), y) = 0$ for all $y \in I$. In the comparative statics analysis, the objective is to determine the sign of:

$$\frac{dx^*}{dy} = \psi'(y_0), \tag{2}$$

which is the instantaneous rate of change of x^* with respect to y_0. Thus, if the sign is positive, then this means that if an infinitesimally small change in the parameter y occurs starting from its initial value y_0, then an increase in the equilibrium value of x^* will also occur. Conversely, if the sign is negative, then an infinitesimally small change in y will produce a reduction in x^*. Moreover, (2) represents a quantitative approximation of the variation produced, and the smaller the variation in the parameter, the more accurate the approximation would be.

In the EU models, the derivative that results in (2) depends on the decision-maker's utility function. Thanks to the Arrow–Pratt measure (Arrow, 1965 and Pratt, 1964), this fact allows the analysis to be performed according to the different types of risk behavior. Nevertheless, due to this dependency on the utility function, the analysis is

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[1] The implicit-function theorem for one dependent variable and one equation can be seen, for instance, in Theorem 3.2.1 in Krantz and Parks (2013).

http://dx.doi.org/10.1016/j.ejor.2015.06.053
0377-2217/

usually approached from a qualitative perspective, i.e., the study focuses on obtaining the sign of the derivative rather than the variation magnitude. In addition, dependency on the utility function, which is absolutely essential for the theoretical analysis of the model, makes it difficult for the model to be applied to empirical studies. The reason for this difficulty is that it becomes necessary to make a number of assumptions related to the decision-maker's utility function, such as the type of function and the parameters involved. Although, in practice, different types of functions have been used, the majority of authors tend to use a decision-maker with a utility function that exhibits decreasing absolute risk aversion (DARA), as stated by Pratt (1964) and Arrow (1965). The most frequently used set of functions in the literature is that of the power utility functions, which exhibits DARA and constant relative risk aversion (CRRA). However, even if we assume a utility function of this type, it is still necessary to make assumptions on the parameter involved (the degree of relative risk aversion). Certain authors, such as Arrow (1971), Friend and Blume (1975), Hansen and Singleton (1982; 1984), Mehra and Prescott (1985) and Brown and Gibbons (1985), have made estimations of this parameter and have obtained different ranges for each case (a summary of some of these results, and others, can be seen in Aït-Sahalia & Lo, 2000). To avoid the problem of implementation of the utility function, other authors have used methods, such as mean-variance (MV) analysis, which is a good approximation of EU maximization (Markowitz, 2014).

Thus, under the assumption of EU maximization, in general, it is difficult to measure the effect of the variation of a parameter because of its dependency on the utility function. Nevertheless, if the simultaneous variation of two parameters is considered, either this effect or the part of the effect depending on the utility function of one of the two parameters may be compensated by the presence of the other. Such research, which has yet to be conducted, would allow both theoretical and empirical conclusions to be obtained with no need to make any assumptions on the utility function and its parameters involved. In addition, not only would a qualitative assessment of the change in the equilibrium value be made possible, but also its quantitative assessment. In this paper, this analysis is performed by using a general model that includes a number of important models in the literature.

The paper is structured as shown below. In Section 2, we show the comparative statics analysis that allows the effect of the simultaneous variation of two or more parameters to be studied. In Section 3, a general decision model under EU maximization is presented. In Section 4, we develop a methodology that allows us to attain comparative statics effects, which are independent of the utility function. In Section 5, we include a number of applications of the methodology used in three different models in the literature, and the most relevant effects attained are analyzed. In Section 6, we numerically illustrate some of these effects. Finally, concluding remarks are made in Section 7.

2. Comparative statics under simultaneous changes in parameters

In this section, the comparative statics analysis, formalized by Samuelson (1941) to study the effect produced on the equilibrium value due to the variation of one single parameter, is extended in order to study the effect of the simultaneous variation of two or more parameters.

First, we assume the following continuously differentiable implicit relation involving an unknown variable x and n parameters $(y_1, \ldots, y_n)$:

$$\phi(x, y_1, \ldots, y_n) = 0. \tag{3}$$

If, for given values of $y_1 = y_{01}, \ldots, y_n = y_{0n}$, there exists a solution x^* to (3), and if $\phi_x(x^*, y_{01}, \ldots, y_{0n}) \neq 0$, then by the implicit-function theorem there exists a continuously differentiable function ψ in a sufficiently small neighborhood I of $(y_{01}, \ldots, y_{0n})$ such that $x^* = \psi(y_{01}, \ldots, y_{0n})$ and $\phi(\psi(y_1, \ldots, y_n), y_1, \ldots, y_n) = 0$ for all $(y_1, \ldots, y_n) \in I$. Therefore, the total differential of x^* is given by:

$$dx^* = \frac{\partial x^*}{\partial y_1} dy_1 + \cdots + \frac{\partial x^*}{\partial y_n} dy_n. \tag{4}$$

Second, our goal now is to study the change in the value of x^* that results when $(y_1, \ldots, y_n)$ varies from the initial position $(y_{01}, \ldots, y_{0n})$ to a new position $(y_{01} + dy_1, \ldots, y_{0n} + dy_n)$, with $dy_i \in \mathbb{R}$ $(1 \le i \le n)$. This change is denoted by Δx^*. That is:

$$\Delta x^* = \psi(y_{01} + dy_1, \ldots, y_{0n} + dy_n) - \psi(y_{01}, \ldots, y_{0n}). \tag{5}$$

It is known that, since ψ is differentiable, the following relationship between (4) and (5) exists:

$$\Delta x^* = dx^* + o(v), \tag{6}$$

where we denote $v \equiv (dy_1, \ldots, dy_n) \in \mathbb{R}^n$ and $o(v)$ is an infinitesimal of a higher order than the norm of the vector of increments, $\|v\|$. Therefore, by neglecting the second term in (6), the following linear approximation is obtained:

$$\Delta x^* \approx dx^*. \tag{7}$$

Therefore, when simultaneous changes occur in the parameters $y_1, \ldots, y_n$, given by $dy_1, \ldots, dy_n$, (4) represents an approximation of the total change occurring in x^*. Thus, the objective in comparative statics under simultaneous changes in parameters becomes the determination of the sign and/or value of (4).

3. A general model of decision-making under risk

In order to conduct a study that encompasses the largest number of models, we will adopt a general formulation of the problem similar to that proposed by Feder (1977). Let us consider an agent facing some type of risk modeled by a random variable $Z = \mu + \sigma\epsilon$, where ϵ is a nondegenerate random variable with $\mathsf{E}[\epsilon] = 0$ and $\mathsf{Var}[\epsilon] = 1$, so that $\mathsf{E}[P] = \mu$ and $\mathsf{Var}[P] = \sigma^2$. This risk affects the agent's future wealth, given by:

$$W(x) = f(x)Z - g(x), \tag{8}$$

where f and g are (nonrandom) functions depending on a decision variable x, with $f_x \neq 0$. In addition, f and g depend on an n-vector of parameters $\mathbf{y} = (y_1, \ldots, y_n)$. In order to simplify, we use notations $f(x)$ and $g(x)$ instead of $f(x, \mathbf{y})$ and $g(x, \mathbf{y})$. The agent must make a decision on the value of x, so as to maximize the EU for this future wealth.

The agent's attitude towards risk can be modeled by a Bernoulli utility function u, which is sufficiently regular (at least, of class $\mathcal{C}^2$), such that $u' \neq 0$ and $u'' \neq 0$. Thus, the agent's goal is to solve:

$$\max_x \mathsf{E}\Big[u\big(W(x)\big)\Big]. \tag{9}$$

The first-order condition is:

$$\mathsf{E}\Big[u'(W)\,(f_x(x)Z - g_x(x))\Big] = 0. \tag{10}$$

If we assume that the second-order condition is verified so that there is an optimum x^* for problem (9), then condition (10) evaluated at this optimum is equal to:

$$F(x^*) = h(x^*), \tag{11}$$

where we denote $h \equiv g_x/f_x$, and:

$$F(x) \equiv \frac{\mathsf{E}[u'(W(x))Z]}{\mathsf{E}[u'(W(x))]}. \tag{12}$$

From now on, we will use notation $W^* \equiv W(x^*)$.

In the literature, there are numerous models with this general structure, most of which are related to the field of the firm under uncertainty. The first model with this structure that we have found