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Short Communication

A closed-form solution for a tollbooth tandem queue with two heterogeneous servers and exponential service times

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ABSTRACT

This paper considers a tollbooth queueing system where customers arrive according to a Poisson process and there are two heterogeneous servers of exponential service times. We show that eigenvalues can be found explicitly for the characteristic matrix polynomial associated with the Markov chain characterizing the system. We derive a closed-form solution for the steady state probabilities to make the straightforward computation of performance measures.

1. Introduction

Tollbooth queues have been studied for a long time, which is motivated by the performance evaluation of practical scenarios such as toll collection systems (Hall & Daganzo, 1983), gas stations (Teimoury, Yazdi, Haddadi, & Fathi, 2011), fast food service to car travelers.

In the recent work (He & Chao, 2014) of He and Chao, based on the review of the relevant literature it is stated that there was no a closed-form solution for the stationary distribution of a tollbooth queue where customers arrive according to a Poisson process and two identical servers of exponential service times.

In this paper we consider the class of tollbooth queues with two *heterogeneous servers*. It is shown that the Markov chain characterizing the queue is of GI/M/1-type. We apply the spectral expansion method (Chakka & Do, 2007; Mitrani & Chakka, 1995) and eigenvalues can be found explicitly for the characteristic matrix polynomial. We derived *a closed-form* solution for the steady state probabilities of this queue class even if two servers are heterogeneous. To our best knowledge, this is the first work that provides a closed-form solution for the problem in question. Numerical results are presented to investigate the operation of a tollbooth system.

The rest of this paper is organized as follows. In Section 2, we show how to use eigenvalues to solve the equilibrium equations. In Section 3, we derive a closed-form solution. In Section 4 numerical results are presented. Finally, Section 5 concludes the paper.

2. Review of the spectral expansion method

Quasi birth–death (QBD) processes, as a generalization of the classical birth and death M/M/1 queues, were first introduced by Evans (1967) and Wallace (1969). The states of a QBD process are described by two dimensional random variables called a phase and a level (Latouche & Ramaswami, 1999; Mitrani & Chakka, 1995; Neuts, 1981). It is worth emphasizing that transitions in a QBD process are only possible between adjacent levels. QBD processes can be used for the performability analysis of many problems in telecommunications and computer networks (Chakka & Do, 2007).

There are two main methods to find the steady state probabilities for QBD processes on semi-infinite strips. The matrix-geometric (Neuts, 1981) method recursively computes the rate matrix (the minimal nonnegative matrix solution) of the matrix quadratic equation. The spectral expansion method computes the eigenvalues and eigenvectors of the characteristic matrix polynomial (Mitrani & Chakka, 1995) to obtain the expressions for the stationary distribution. Mitrany and Avi-Itzhak (1968), and Bertsimas (1990) used eigenvalues for the computation of the steady state probabilities. Applying the spectral expansion method, Grassmann (2003); Grassmann and Drekic (2000); Grassmann and Tavakoli (2005) showed that there exists an efficient algorithm to find the eigenvalues of the matrix quadratic equation of a tridiagonal form based on the sign variations of the Sturm sequences. Do (2010) showed that there exists a closed-form for eigenvalues in the M/M/1 retrial queue with working vacations.

Markov chains of M/G/1-type (skip-free to the left) and Markov chains of GI/M/1-type (skip-free to the right) can be solved within the

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QBD framework using a transformation where a three-dimensional process is represented by a special QBD with a two dimensional phase (Latouche & Ramaswami, 1999). It worth mentioning that matrix analytic methods were developed by Neuts (1981) for Markov chains of M/G/1-type and Markov chains of GI/M/1-type as well. Chakka and Do (2007) applied the spectral expansion method for Markov chains where one-step transitions are allowed across several levels. In Section 3 we will show that a Markov chain characterizing a tollbooth queue with two heterogeneous servers is of GI/M/1-type.

3. A closed-form solution

3.1. Notations and assumptions

We consider a tollbooth queue that consists of two servers. This kind of queueing systems can be used to model a toll system where two service people collect payments from car travelers in sequentially placed booths. There is a buffer of infinite size in the front of the first server and there is no waiting area between two servers. Customers arrive according to a Poisson process with rate λ. Service times are exponentially distributed with parameter μ_1 at server 1 and parameter μ_2 at server 2.

A tollbooth queue is characterized by the following special service rule:

- Each customer only requests service from one server. Two servers can be simultaneously idle if there is no customer waiting in the buffer.
- The first in first out (FIFO) principle is applied regarding the service order of customers in each server.
- A customer who finishes his/her service at the first server cannot depart from the system if the second server is busy. This customer is blocked to depart and must wait until the departure of another customer being served by the second server (then two customers leave the system together). In this case, the first server is idle, but the first server could not serve a customer waiting in the front of the buffer because the service position is occupied by the "blocked" customer.
- If the second server is idle after the departure of a certain customer and the first server is busy with another customer, the second server could not start the service of a customer waiting in the buffer. In this case, the customer being served by the first server blocks the movement of customers from the buffer.
- If two servers are idle, an arriving customer is served by the second server.

Let $J(t)$ denote the number of customers in the system at time instant t. Let $I^*(t)$ be the state of servers as in Table 1. The system is described by continuous time Markov chain (CTMC) $\{I^*(t), J(t)\}$ on state space $S = \{(0,0) \cup (1,1) \cup (2,1) \cup (k,l) : \forall k = 1,2,3 \text{ and } l \geq 2\}$. The dynamics of CTMC is driven by the arrivals and the departures of customers. From the operation rules, the following possibilities can happen upon the arrival of a specific customer.

Table 1
The states of the servers.

$I^*(t)$	Server 1	Server 2
0	Idle	Idle
1	Idle	Busy
2	Busy	Idle
3	Busy	Busy

- If $J(t) = 0$, the customer obtains service from the second server.
- If $J(t) = 1$ and the first server is busy, the customer waits in the buffer.
- If $J(t) = 1$ and the second server is busy, the customer is served by the first server.
- If $J(t) \geq 2$, the customer enters the buffer.

As a consequence, the transition rate diagram can be depicted in Fig. 1.

3.2. Balance equations

We denote the steady state probabilities by

$$\pi_{i,j} = \lim_{t\to\infty} \Pr(I^*(t) = i, J(t) = j), \quad (i,j) \in S.$$

The balance equations can be written as follows

$$\pi_{0,0}\lambda = (\pi_{1,1} + \pi_{1,2})\mu_2 + \pi_{2,1}\mu_1, \tag{1}$$

$$\pi_{1,1}(\mu_2 + \lambda) = \pi_{0,0}\lambda + \pi_{1,3}\mu_2 + \pi_{2,2}\mu_1, \tag{2}$$

$$\pi_{2,1}(\mu_1 + \lambda) = \pi_{3,2}\mu_2, \tag{3}$$

$$\pi_{1,2}(\mu_2 + \lambda) = \pi_{3,2}\mu_1, \tag{4}$$

$$\pi_{2,2}(\mu_1 + \lambda) = \pi_{2,1}\lambda + \pi_{3,3}\mu_2, \tag{5}$$

$$\pi_{3,2}(\mu_2 + \mu_1 + \lambda) = \pi_{1,1}\lambda + \pi_{2,3}\mu_1 + \pi_{1,4}\mu_2, \tag{6}$$

$$\pi_{1,j}(\mu_2 + \lambda) = \pi_{1,j-1}\lambda + \pi_{3,j}\mu_1, \ j \geq 3, \tag{7}$$

$$\pi_{2,j}(\mu_1 + \lambda) = \pi_{2,j-1}\lambda + \pi_{3,j+1}\mu_2, \ j \geq 3, \tag{8}$$

$$\pi_{3,j}(\mu_2 + \mu_1 + \lambda) = \pi_{3,j-1}\lambda + \pi_{2,j+1}\mu_1 + \pi_{1,j+2}\mu_2, \ j \geq 3. \tag{9}$$

The normalization equation is

$$\pi_{0,0} + \pi_{1,1} + \pi_{2,1} + \sum_{j=2}^{\infty}(\pi_{1,j} + \pi_{2,j} + \pi_{3,j}) = 1. \tag{10}$$

Let us introduce

$$A_0 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

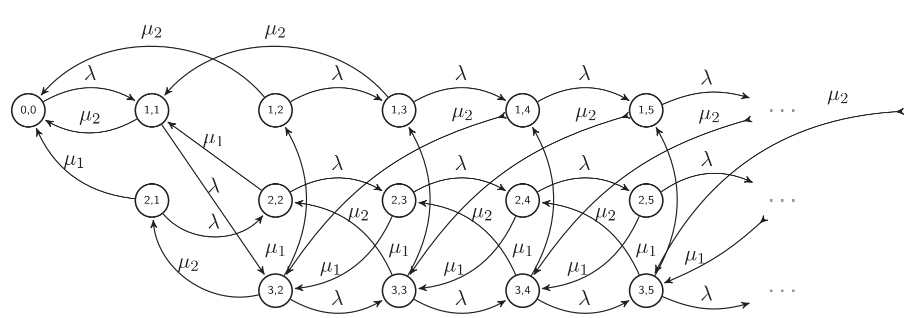

Fig. 1. The transition rate diagram of a Markov process.