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Discrete Optimization

A branch-cut-and-price algorithm for the piecewise linear transportation problem

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ABSTRACT

In this paper we present an exact solution method for the transportation problem with piecewise linear costs. This problem is fundamental within supply chain management and is a straightforward extension of the *fixed-charge transportation problem*. We consider two Dantzig–Wolfe reformulations and investigate their relative strength with respect to the linear programming (LP) relaxation, both theoretical and practical, through tests on a number of instances. Based on one of the proposed formulations we derive an exact method by branching and adding generalized upper bound constraints from violated cover inequalities. The proposed solution method is tested on a set of randomly generated instances and compares favorably to solving the model using a standard formulation solved by a state-of-the-art commercial solver.

1. Introduction

In this paper we consider the problem of finding a minimum cost flow in a bipartite graph between a set of suppliers and a set of customers. The cost of sending goods on an arc follows a piecewise linear structure (see Section 2) and the problem is thereby a natural generalization of the *Fixed-Charge Transportation Problem*. This problem is termed the *Piecewise Linear Transportation Problem* (PLTP), and is a versatile problem that is fundamental within supply chain network design and arises in a number of applications. The general form of the cost functions allows for modeling of different transportation modes such as *small packages, less-than-truckloads, truckloads,* and *air freight* (see e.g. Croxton, Gendron, & Magnanti, 2003b; Lapierre, Ruiz, & Soriano, 2004). Additionally, the cost function can be used to model price discounts such as all-unit or incremental discounts, often found in procurement theory (see Davenport & Kalagnanam, 2001; Kameshwaran & Narahari, 2009) or to linearize an otherwise nonlinear cost function.

Kim and Pardalos (2000) present a heuristic for the PLTP based on a linearization of the cost function and subsequent solution of a (standard) transportation problem. In Croxton, Gendron, and Magnanti (2003a) the authors show that the linear programming relaxations of three textbook formulations of a piecewise linear function are equivalent. One of them is the *Multiple-Choice Model* (MCM) used in Section 2.1. Other studies (e.g. Keha, de Farias, & Nemhauser, 2004; Vielma, Ahmed, & Nemhauser, 2008, 2010) have extended this result to include a number of other formulations and they also perform tests to find the best formulation in terms of solving the problem to optimality by a standard solver. The most recent of these studies suggests that when the number of different transportation modes is relatively small (as in our tests), the MCM, presented in Section 2.1, is preferable. As the linear programming relaxation of the standard models is often very poor, we propose two stronger formulations for the problem, both based on a Dantzig–Wolfe reformulation of the problem.

In Section 2 we give a formal definition of the problem using the standard multiple-choice formulation and two new formulations. The two new stronger formulations rely on a Dantzig–Wolfe reformulation of the original problem and column generation is required to solve the LP relaxation. The strength of the linear programming relaxation of the formulations is investigated in Section 3, along with the computational experience on a test bed of instances. Based on these results we propose an exact solution method based on one of the formulations, in which we add valid inequalities described in Section 4 and by applying the branching rule described in Section 5. In Section 6 we test the solution method on a number of randomly generated instances and compare the method to solving the MCM by a standard commercial solver. Section 7 summarizes our findings and concludes this paper.

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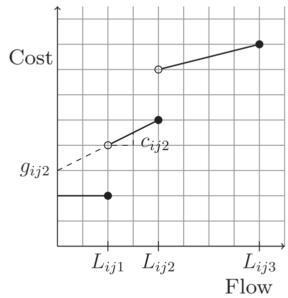

Fig. 1. A piecewise linear cost function.

2. Mathematical formulations

In this section we first define the PLTP and introduce notation. Then we introduce a standard formulation and two new formulations of the PLTP.

Let the set of supply nodes (suppliers) be denoted by the set $I = \{1, \ldots, n\}$. The total capacity of each supplier i is denoted by S_i. The demand nodes (customers) are denoted by the set $J = \{1, \ldots, m\}$, where customer j has demand d_j. The cost of transporting goods from supplier $i \in I$ to customer $j \in J$ follows a piecewise linear cost structure with κ_{ij} line segments on the arc between supplier i and customer j, which is also known as the modes. For notational convenience we will assume that $\kappa_{ij} = \kappa$ for all (i, j) and we denote the set of modes by Q. Each mode $q \in Q$ from i to j is characterized by a fixed cost for using the mode, g_{ijq} and a variable cost (the slope of the mode), c_{ijq}. Additionally, the flow using mode q on the arc (i, j) is restricted to a minimum of $L_{ij,q-1}$ and a maximum of L_{ijq} (see Fig. 1), where $L_{ij0} = 0$. We assume that $L_{ij\kappa} \leq \min\{S_i, d_j\}$, i.e. that the maximum flow between supplier i and customer j does not exceed neither the capacity S_i nor the demand d_j, respectively. Note that the maximum capacity on an arc might be restrictive, i.e. the inequality might be strict. Hence, formally we consider a separable, lower-semicontinuous, piecewise linear function with κ line segments defined on the interval from 0 to $L_{ij\kappa}$ (see Vielma et al., 2010 for more on other kinds of piecewise linear functions).

2.1. The multiple-choice model

One standard way of representing a discontinuous, piecewise linear function is by the so-called *Multiple-Choice Model*. Using this formulation, the problem can be stated as

$$\text{(MCM)} \quad \min \sum_{i\in I}\sum_{j\in J}\sum_{q\in Q}(c_{ijq}x_{ijq} + g_{ijq}v_{ijq}), \tag{1}$$

$$\text{s.t.} \sum_{i\in I}\sum_{q\in Q} x_{ijq} = d_j, \quad \forall j \in J, \tag{2}$$

$$\sum_{q\in Q} v_{ijq} \leq 1, \quad \forall (i,j), i \in I, j \in J \tag{3}$$

$$\sum_{j\in J}\sum_{q\in Q} x_{ijq} \leq S_i, \quad \forall i \in I, \tag{4}$$

$$x_{ijq} \leq L_{ijq}v_{ijq}, \quad \forall (i,j,q), i \in I, j \in J, q \in Q \tag{5}$$

$$x_{ijq} \geq L_{ij,q-1}v_{ijq}, \quad \forall (i,j,q), i \in I, j \in J, q \in Q \tag{6}$$

$$x_{ijq} \geq 0, \quad \forall (i,j,q), i \in I, j \in J, q \in Q \tag{7}$$

$$v_{ijq} \in \{0,1\}, \quad \forall (i,j,q), i \in I, j \in J, q \in Q. \tag{8}$$

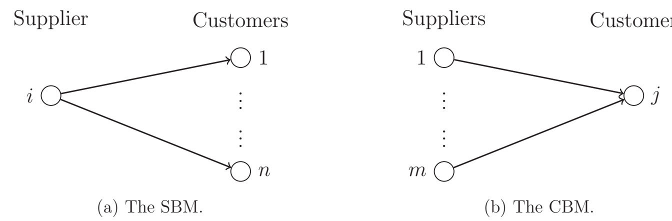

Fig. 2. The basic idea for each reformulation.

The objective, (1), is to minimize the total costs of supplying the customers. Each customer has to receive an amount equal to its demand by (2). Constraints (3) enforce that at most one mode is used between each combination of supplier and customer. Constraints (4) state that the solution has to obey the capacity at each supplier. Constraints (5) and (6) bound the flow within the associated upper and lower bounds for mode q between the supplier i and customer j, respectively. Variable x_{ijq} represents the flow between i and j on the mode q and the associated binary variable v_{ijq} is 1 if this mode is used, 0 otherwise. The linear programming relaxation of the MCM is equivalent to replacing the objective function by its lower convex envelope (see e.g. Croxton et al., 2003a). We denote by MCM-LP the relaxed problem defined by the program (1)–(8), where (8) is replaced by the constraints $0 \leq v_{ijq} \leq 1$, $\forall (i, j, q), i \in I, j \in J, q \in Q$.

The two new formulations are based on the introduction of variables for each feasible flow-vector from either each supplier to all the customers, or from all suppliers to each customer (see Fig. 2). These models are denoted the *Supplier-Based Model* (SBM) and the *Customer-Based Model* (CBM), respectively. Both formulations are obtained by applying a Dantzig–Wolfe reformulation of the MCM, while keeping either the demand constraints (2) (for the SBM) or the supply constraints (4) (for the CBM) in the master problem.

2.2. The supplier-based model

Let χ_i denote the set of all feasible flows from supplier i to the m customers with an accumulated flow less than or equal to S_i and the flow on each arc (i, j) less than or equal to $L_{ij\kappa}$, i.e. $\chi_i := (x_{ij})_{j\in J} : 0 \leq x_{ij} \leq L_{ij\kappa}, \forall j \in J$ and $\Sigma_{j\in J} x_{ij} \leq S_i\}$ and T_i denote the set of indices of elements of χ_i. That is, $x_i^t \in \chi_i$, for $t \in T_i$, is a vector with m entries and each entry x_{ij}^t represents the flow on the arc (i, j). For each flow index $t \in T_i$, we define the associated costs of the flow by C_i^t and a binary variable β_i^t, which is equal to one if the flow characterized by the vector t is used and zero otherwise. Now, the PLTP can be formulated as

$$\text{(SBM)} \quad \min \sum_{i\in I}\sum_{t\in T_i} \beta_i^t C_i^t, \tag{9}$$

$$\text{s.t.} \sum_{t\in T_i} \beta_i^t = 1, \quad \forall i \in I, \tag{10}$$

$$\sum_{i\in I}\sum_{t\in T_i} \beta_i^t x_{ij}^t = d_j, \quad \forall j \in J, \tag{11}$$

$$\beta_i^t \in \{0,1\}, \quad \forall i \in I, \forall t \in T_i, \tag{12}$$

The objective is to minimize the total cost as defined in Eq. (9). Constraints (10) state that for each supplier, exactly one flow in T_i must be chosen, assuming that the all-zero flow belongs to T_i. The demand of each customer must be satisfied by constraints (11). Finally, all variables β_i^t must be binary.