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Decision Support

Cost sharing solutions defined by non-negative eigenvectors

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ABSTRACT

The problem of sharing a cost M among n individuals, identified by some characteristic $c_i \in \mathbb{R}_+$, appears in many real situations. Two important proposals on how to share the cost are the *egalitarian* and the *proportional* solutions. In different situations a combination of both distributions provides an interesting approach to the cost sharing problem. In this paper we obtain a family of (*compromise*) solutions associated to the Perron's eigenvectors of Levinger's transformations of a characteristics matrix A. This family includes both the *egalitarian* and *proportional* solutions, as well as a set of suitable intermediate proposals, which we analyze in some specific contexts, as claims problems and inventory cost games.

1. Introduction

A (cost-surplus) *sharing problem* consists of the division of a certain amount M among a group of n agents. We assume that each agent i ($i = 1, 2, \dots, n$) is identified by means of a (numerical) *characteristic* c_i relevant in the distribution of the total amount M. We may find many real situations covered by this simple model. Let us observe, for instance, the following ones:

[1] A typical example is how to allocate the total cost M of a facility (for instance, *a water supply system*) among the n towns that will share it. The characteristic c_i may be, in this example, the proportion of the total population living in each of these towns. A *cost allocation* is a vector $x = (x_1, x_2, \dots, x_n)$ such that $x_1 + x_2 + \dots + x_n = M$. The two more popular ways to share the total cost M are the egalitarian and the proportional ones (see, for instance, Moulin, 1988).

[2] Another example comes from analyzing costs in one-to-many distribution systems. This kind of problem has a logistic feature, namely the *distribution cost* (see, for instance, Turkensteen & Klose, 2012). A single facility, whose location is yet to be determined, serves geographically dispersed demand points (*consumers*). Once the location of the facility and the *distribution cost* (the cost of the optimal delivery route) M has been obtained, the problem to be analyzed is how the cost M is allocated to consumers. Again, there are two *focal* positions: the cost is allocated in an egalitarian way, so any costumer pays (*per unit*) an equal part of the total distribution cost; or, the cost is allocated in a proportional way with respect to the characteristic c_i, that in this example could represent the *distance* between the facility and the consumer i, or the cost of a *direct delivery from the facility to the consumer*.

[3] The so-called *claims problem* (O'Neill, 1982) involves n agents identified by some claims c_i, $i = 1, 2, \dots, n$ on an estate M that is not sufficient to cover the aggregate claim $C = \sum_{i=1}^{n} c_i$, that is $C > M$. The problem is how to allocate the estate M among the claimants by taking into account the claims c_i the agents have on it. Among the many rules defined in this class of problems, two prominent solutions are the proportional and the egalitarian ones.

[4] In a recent paper, Karsten and Basten (2014) present a model on how to share the benefits in pooling of spare parts between multiple users. As they mention, "important savings may be obtained if pooling is taken into account. While promising this does raise the question on how the [users] should share the benefits". In order to construct their model, they use the proportional rule to divide the costs (in this case, proportionality is with respect to the demand rate of each individual). Instead of using proportionality, an egalitarian allocation could also be considered. A similar situation may be found in Meca, Timmer, García-Jurado, and Borm (2004) (see also Fiestras-Janeiro, García-Jurado, Meca, & Mosquera, 2011) where an *inventory cost game* is presented in order to allocate the benefits of cooperation. They use a proportional approach in order to define a solution that is proved to be on the core of the cooperative game.

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All of the above situations may be expressed by the abstract model:

> An amount M must be shared among agents $i = 1, 2, \dots, n$ by considering the individuals' characteristics c_i.

As mentioned, the *proportional* and the *egalitarian* solutions are two focal approaches to solve this model. In many situations, a combination of these solutions could make sense. For instance, if we go back to the water supply system example, it is clear that the population (the *characteristic*) of each town affects the size of the system. Then, a proportional (*with respect to the characteristics*) distribution of the cost seems reasonable. But, on the other hand, most of the cost does not depend on the supplied population, so an egalitarian share (*regardless of the characteristics*) could be as reasonable as the proportional division. As consequence, a combination of both distributions provides an interesting approach. Our proposal is within this perspective.

One possibility is to consider *convex combinations* of these two solutions. This idea has been recently developed in the context of *claims problems* (situation [**3**]) where the characteristic c_i represents the individual *i*s claim (see Giménez-Gómez & Peris, 2014). In that article they propose the $\alpha_{\min}$ solution by using a specific convex combination of the egalitarian and the proportional solutions with the idea of guaranteeing a minimal amount to each individual (regardless of their claims) and allocating the remaining in a proportional way.

We propose an alternative approach to obtain intermediate solutions: instead of taking convex combinations of the *egalitarian* and *proportional* solutions, we use eigenvectors of some suitable matrices. In so doing, we define the *characteristics matrix* of a problem as the matrix whose rows are all identical to the vector of individuals' characteristics c_i. By using this matrix A, we get an alternative way of obtaining the main solutions. In particular, the *egalitarian* solution corresponds to the right eigenvector of matrix A and the *proportional* solution corresponds to the left eigenvector (or, equivalently, to the right eigenvector of the transpose matrix A^t). Then, we present our family of *compromise solutions* for the sharing problem, as the ones associated to the Perron's eigenvectors of each convex combination of the characteristics' matrix and its transpose (Levinger's transformation). The *egalitarian* and *proportional* solutions are obtained for $\alpha = 0$ and $\alpha = 1$, respectively. We analyze the main properties of these compromise solutions as functions of the parameter α that defines the convex combination.

When analyzing some particular situations, we realize that the egalitarian solution could propose inadmissible allocations. For instance, in situation [**2**], if c_i represents the cost of the individual delivery, *rationality* implies that no agent should pay more than this amount. In situation [**3**] the characteristic c_i denotes the claim that individual i has on the endowment. Then, it is obvious that no agent should receive more than her claim. Finally, in situation [**4**] the proposal should give an allocation in the core of the cooperative game. As in the egalitarian case, other solutions associated to Perron's eigenvectors of Levinger's transformations could be inadmissible proposals. However, we will show that it is always possible to find some admissible ones.

The paper is organized as follows. Section 2 introduces the mathematical model we are going to use, and defines the *characteristics matrix* associated to a cost sharing problem. In Section 3 we obtain the main properties of eigenvalues and eigenvectors of the convex combinations of the characteristics matrix and its transpose. Finally, in Section 4, we analyze some particular situations (*claims problems* and *inventory cost games*).

2. Main definitions

2.1. Cost sharing problems

A *cost sharing problem with relevant characteristics* is defined by a finite set of agents $N = \{1, 2, \dots, n\}$, the total cost M to be allocated to the agents, and the characteristics vector[1] $c^t = [c_1, c_2, \dots, c_n]$, identifying each agent. The issue is how to share the total cost M among the agents.[2]

A *cost sharing solution* (*sharing rule*) associates to each sharing problem (M, c), an allocation $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, such that $x_i \geq 0$ (no subsidies are allowed) and $\sum_{i=1}^{n} x_i = M$ (efficiency). The value x_i is the amount individual i contributes to the total cost M.

The most popular sharing rules are the *egalitarian* (*Egal*) and the *proportional* (*Prop*) ones. Equal division establishes that each agent is allocated the same part of the total amount,

$$x_i = Egal_i \equiv \frac{M}{n}, \quad \text{for all} \quad i \in N.$$

Proportional division, with respect to vector c establishes the cost allocation

$$x_i = Prop_i \equiv \frac{c_i}{\sum_{i=1}^{n} c_i} M, \quad \text{for all} \quad i \in N.$$

If we denote by v_{eg} the vector with all entries equal to $1/n$, and by v_{pr} the vector whose i-component is $\frac{c_i}{\sum_{i=1}^{n} c_i}$, then the egalitarian and proportional solutions may be written as:

$$Egal = Mv_{eg}, \quad Prop = Mv_{pr}.$$

2.2. Matrix analysis

Given a square *positive* matrix A ($a_{ij} > 0$, for all i, j), Perron's theorem (Perron, 1907) ensures the existence of an eigenvalue of A, $\lambda(A)$, which is strictly positive and has associated a positive eigenvector $v^* > 0$ such that $Av^* = \lambda(A)v^*$, and $\lambda(A) = \rho(A)$, the spectral radius of this matrix.[3] We consider positive eigenvectors normalized so that the sum of its components is equal to 1, $\sum_{i=1}^{n} v_i^* = 1$. Note that this fact guarantees the uniqueness of the associated positive eigenvector.

Given a positive square matrix A, the Levinger's transformation of A (Levinger, 1970) is the family of matrices obtained throughout convex combinations of A and its transpose A^t, that is, defined by

$$\mathcal{A}(\alpha) = \alpha A^t + (1 - \alpha)A \quad \alpha \in [0, 1].$$

We denote by $v(\alpha)$ the normalized positive eigenvector associated with the dominant eigenvalue $\lambda(\alpha)$, in the corresponding α Levinger's transformation. We know (see Fiedler, 1995) that the Levinger's function, defined by Perron's eigenvalue $\lambda(\alpha)$, is a non-decreasing function in the sub-interval $[0, \frac{1}{2}]$ and that it is symmetric about $\alpha = \frac{1}{2}$.

2.3. From eigenvectors to sharing proposals

Any positive normalized vector $v^* > 0$ provides a way of sharing an amount M among n individuals:

$$x_i = Mv_i^*, \quad i = 1, 2, \dots, n.$$

If we use the normalized eigenvectors $v(\alpha)$, associated to the Levinger's transformation $\mathcal{A}(\alpha)$, then we obtain a family of possible cost allocations, by varying the parameter α,

$$x_i(\alpha) = Mv_i(\alpha), \quad i = 1, 2, \dots, n.$$

[1] Throughout the paper, vectors are considered column vectors, $v_{n\times 1}$. The transposed (row) vector is denoted by v^t.

[2] Although we consider cost sharing problems, our reasoning could be immediately applied to surplus sharing situations.

[3] Moreover, this eigenvalue is simple, strictly greater than any other eigenvalue, and no other eigenvalue has a positive eigenvector associated, see, for instance, Berman and Plemmons (1994).