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Discrete Optimization

An approach to the asymmetric multi-depot capacitated arc routing problem

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ABSTRACT

Despite the fact that the Capacitated Arc Routing Problems (CARPs) received substantial attention in the literature, most of the research concentrates on the symmetric and single-depot version of the problem. In this paper, we fill this gap by proposing an approach to solving a more general version of the problem and analysing its properties. We present an MILP formulation that accommodates asymmetric multi-depot case and consider valid inequalities that may be used to tighten its LP relaxation. A symmetry breaking scheme for a single-depot case is also proposed. An extensive numerical study is carried to investigate the properties of the problem and the proposed solution approach.

1. Introduction

The Capacitated Arc Routing Problem (CARP) originally defined by Golden and Wong (1981) has been extensively studied during the recent decades due to its numerous potential applications, including post delivery, waste collection, winter services (e.g. salt gritting or snow plowing), public transport routing, etc. To fulfil the needs of applications, a number of extensions and modifications of the CARP were considered. These can be roughly classified based on the following features:

- symmetric (undirected) vs. asymmetric (including a particular case of directed) network;
- single vs. multiple vehicles;
- single vs. multiple depots;
- single vs. multiple objectives (see, e.g., Grandinetti, Guerriero, Laganá, & Pisacane, 2003);
- additional constraints: time windows, priorities, etc.

For an overview of the modifications and solution methods we refer the reader to Corberan and Prins (2010); Hertz (2005). The available solution approaches can be divided into two broad categories: (a) those based on a reduction of the ARP to the Vehicle (Node) Routing Problem (VRP) (Longo, de Aragão, & Uchoa, 2006), and (b) those designed specifically for the ARP. The success of the first group of approaches is partially based on the fact that the VRP is substantially better studied than the ARP. At the same time, the transformation of the ARP to the VRP ignores sparsity of the underlying graph and creates computational problems for larger networks. On the contrary, most of the original ARP approaches utilise the sparse nature of realistic road networks (see Fig. 1), which allows large instances at least to be handled, if not solved to optimality (see, e.g., Bartolini, Cordeau, & Laporte, 2013; Bode & Irnich, 2012, 2014; Corberan, Oswald, Plana, Reinelt, & Sanchis, 2012 for a single vehicle case).

As seen from the literature, most research concentrates on the undirected version of the problem (see, e.g., Belenguer & Benavent, 2003; Bode & Irnich, 2014 and references within). At the same time, one way streets are not uncommon, which implies that a realistic approach should be able to handle this case. Furthermore, we could not find an exact approach dealing with the multi-depot CARP – another natural extension of the original problem. These observations motivated us to consider a more general version of the CARP with multiple depots and directed arcs.

The major goal of this paper is to study the properties of the CARP and to consider the possibility of finding exact solutions to the multi-depot CARP for realistic (large, sparse, asymmetric) road networks.

This paper is organised as follows. The next section presents the basic MILP formulation utilised throughout the paper. Section 3 focuses on the inequalities that can be used to tighten the formulation, followed by Section 4 describing our branch-and-cut solution approach. Section 5 provides a computational study of our approach. Finally, Section 6 concludes the paper with a summary of main results and future research directions.

2. The basic MILP model

In this section, we propose a two-index MILP formulation for the asymmetric multi-depot CARP. First, we introduce some notions and notation.

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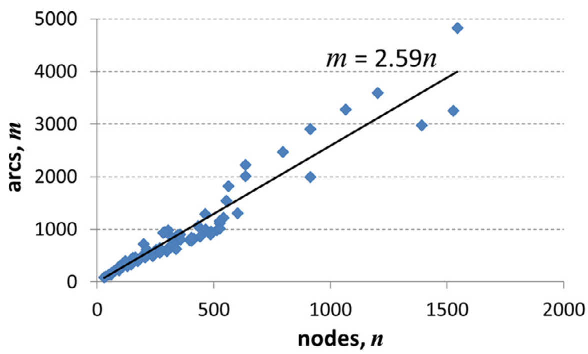

Fig. 1. Sparsity of real road networks for several urban areas in Europe and USA. Clearly, the number of arcs is of order n, i.e. much smaller than the potential limit of n^2 (n is the number of nodes).

2.1. Notions and notations

Through the rest of the paper, let $G(V, A, w)$ denote a directed weighted graph with the set of vertices V ($|V| = n$), the set of arcs A ($|A| = m$) and a weight function $w(\cdot) : A \to \mathbb{R}_+$. Given some set $S \subset V$, $G(V \setminus S)$ denotes a graph obtained from G by deleting each vertex from S together with all incident arcs.

A *walk* of length L in G is a sequence $i_1 - i_2 - \cdots - i_L$ of vertices such that there is an arc between each consecutive pair, i.e. $(i_l, i_{l+1}) \in A$ for all $l = 1, \ldots, L-1$. Without any ambiguity one may think of a walk in terms of the corresponding arcs, rather then vertices. Under a *tour*, we understand a walk for which holds $i_1 = i_L$, and by the weight of a walk we understand the sum of the weights of all arcs in it. Note that the multiplicity of arcs in a walk does matter, i.e. if some arc is traversed three times then its weight contributes to the weight of the walk with a factor of 3. An *elementary cycle* is a tour traversing each vertex and arc at most once. Given a directed cycle $i_1 - i_2 - \cdots - i_j - \cdots - i_k - \cdots - i_L$, arc (i_j, i_k) is a *forward chord* if $i_1 - i_2 - \cdots - i_j - i_k - \cdots - i_L$ is also a directed cycle. Respectively, arc (i_k, i_j) is a *backward chord*.

We say that vertex j is *reachable* from vertex i, if there exists a directed $i - j$ path; arc (j, l) is reachable from i, if j is reachable from i.

Further, for any subset $S \in V$ let us denote by $\delta^+(S) = \{(i, j) \in A \,|\, i \in S, j \in V \setminus S\}$ and $\delta^-(S) = \{(i, j) \in A \,|\, i \in V \setminus S, j \in S\}$ the sets of arcs having one of the endpoints in S; for singletons we use shortcuts $\delta^+(i) = \delta^+(\{i\})$ and $\delta^-(i) = \delta^-(\{i\})$.

2.2. The formulation

The problem under consideration can be formalised as follows. Given a directed weighted graph $G(V, A, w)$, a demand function $d(\cdot) : A \to \mathbb{R}_+$, selected vertices (depot vertices) $v_k^d \in V$ ($k \in K$) and a number $Q \in \mathbb{R}_+$ (capacity), the goal is to find:

- $|K|$ tours of the minimum total weight, each tour traversing a corresponding vertex v_k^d ($k \in K$);
- an assignment of arcs with positive demand to the tours, such that each arc is assigned to one of the tours and the sum of demands assigned to a tour does not exceed Q.

Let us denote $d_a = d(a)$, $A^d = \{a \in A \,|\, d(a) > 0\}$ and introduce two sets of variables: $z_a^k \in \mathbb{Z}_+$ ($a \in A$) denote how many times arc $a \in A$ is traversed by tour k, and $x_a^k \in \{0, 1\}$ ($a \in A^d$) reflect the assignment of arcs to the tours. Now, the problem can be formulated as follows.

$$\sum_{k \in K} \sum_{a \in A} w_a z_a^k \longrightarrow \min \tag{1}$$

s.t.

$$\sum_{a \in A} d_a x_a^k \le Q, \quad k \in K \tag{2}$$

$$\sum_{k \in K} x_a^k = 1, \quad a \in A^d \tag{3}$$

$$\sum_{a \in \delta^+(i)} z_a^k = \sum_{a \in \delta^-(i)} z_a^k, \quad i \in V, k \in K \tag{4}$$

$$x_a^k \le z_a^k, \quad a \in A^d, k \in K \tag{5}$$

$$\sum_{a \in \delta^-(i)} u_a^k \le 1, \quad i \in V, k \in K \tag{6}$$

$$u_a^k \le z_a^k, \quad a \in A, k \in K \tag{7}$$

$$\sum_{a \in \delta^-(v_k^d)} u_a^k = 0, \quad k \in K \tag{8}$$

$$y_{v_k^d}^k = 0, \quad k \in K \tag{9}$$

$$y_j^k \ge y_i^k + 1 + M(u_{(i,j)}^k - 1), \quad (i, j) \in A, k \in K \tag{10}$$

$$\sum_{a \in \delta^-(i)} u_a^k \ge \frac{1}{M} \sum_{a \in \delta^+(i)} z_a^k, \quad i \in V \setminus \{v_k^d\}, k \in K \tag{11}$$

$$z_a^k \in \mathbb{Z}_+, \quad a \in A, k \in K \tag{12}$$

$$x_a^k \in \{0, 1\}, \quad a \in A^d, k \in K \tag{13}$$

$$y_i^k \in \mathbb{R}_+, \quad i \in V, k \in K \tag{14}$$

$$u_a^k \in \{0, 1\}, \quad a \in A, k \in K \tag{15}$$

Objective (1) explicitly minimises the total weight of all the tours. The capacity of each tour is limited by constraints (2), while constraints (3) ensure that each demand arc is served. Constraints (4) are flow conservation constraints. Further, constraints (5) ensure that only traversed demand arcs can be assigned to a tour. Finally, constraints (6)–(11) ensure connectedness of each tour by constructing a tree rooted at the depot vertex v_k^d and spanning all vertices traversed by a tour. The tree is oriented so that there is a directed path from the root to each leaf. Variables u_a^k reflect the arcs in such a tree, while variables y_i^k denote the level of vertex i in a tree. The root has level 0 (as assigned by constraints (9)), its neighbours have level 1, etc. Constraints (6) and (7) ensure that each vertex in the tree has at most one incoming arc and that the tree contains only the traversed arcs, respectively. Constraints (10) prohibit cycles. Constraints (11) ensure that each traversed vertex, except the depot, has an incoming arc in the tree. Parameter M in (10) and (11) is a large enough positive number. Note that constraints (6)–(11) are given here purely on the purpose of making the formulation complete and are not used further in this paper.

It is not hard to understand that the proposed formulation can be adjusted for the case of an undirected graph by assuming that each undirected edge is represented by two arcs and replacing constraints (3) by:

$$\sum_{k \in K} x_{(i,j)}^k + x_{(j,i)}^k = 1, \quad (i, j) \in A^d. \tag{16}$$

Though this formulation has a polynomial size, it is quite extensive, both in terms of constraints ($|A^d| + (4 + 3n + 2|A^d| + 5m)|K|$) and variables ($m|K| + |A^d| \cdot |K| + n|K| + m|K|$), $2m|K| + |A^d| \cdot |K|$ of which are integer. In case all arcs have positive demands, $A^d = A$ holds and these quantities become $n + (4 + 3n + 7m)|K|$, $(n + 3m)|K|$ and $3m|K|$, respectively. However, it can be seen that as much as $(3 + 3n + 5m)|K|$ constraints and $(n + m)|K|$ variables ($m|K|$ of which are integer) are used purely to guarantee connectedness of the tours. In fact, constraints (6)–(11) can be replaced by the following subtour eliminating constraints:

$$\sum_{a \in \delta^+(S)} z_a^k \ge \frac{1}{M} \sum_{a \in A'} z_a^k, \quad k \in K, \tag{17}$$